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SOME TOPICS IN SAMPLING THEORY*

BY H. L. RIETZ

1. *Introduction.* Many important contributions have been made to "small or exact sampling theory" since 1908, when "Student's" contribution† created a new interest in the subject. A general notion of the extent of the literature on this topic may be obtained by an examination of the excellent survey by P. R. Rider‡ in 1930. Next, J. O. Irwin§ gave valuable reports on this literature for approximately the years 1930–1934. Then Rider‖ brought the survey on exact sampling theory well up-to-date in a paper of 1935, and again for part of the field in 1936.

The invitation to give a paper at this meeting left me free as to the selection of a subject. This freedom was interpreted to mean that you would probably prefer to have me speak on some topics in which I have a special interest, rather than to attempt a well-balanced discussion of recent progress for which we may well turn to the papers mentioned above as surveying the contributions to exact sampling theory.

On the side of applications, sampling theory is much concerned with judging, by means of one or more tests, whether an observed random sample, taken as a whole, conforms reasonably to samples expected from a specified population. Test criteria may be based on such concepts as the mean, the variance, the standard deviation, the Pearson χ^2, the "Student" ratio, the Fisher z, the correlation ratio, and other statistical estimates of

* An address delivered before the Society on November 27, 1936, in Lawrence, by invitation of the Program Committee.

† *The probable error of the mean*, Biometrika, vol. 6 (1908–09), pp. 1–25.

‡ *A survey of the theory of small samples*, Annals of Mathematics, (2), vol. 31 (1930), pp. 577–628.

§ *Recent advances in mathematical statistics*, Journal of the Royal Statistical Society, vol. 94 (1931), pp. 568–578; vol. 95 (1932), pp. 498–530; vol. 97 (1934), pp. 114–154; vol. 99 (1936), pp. 714–769.

‖ *Recent progress in statistical method*, Journal of the American Statistical Association, vol. 30 (1935), pp. 58–88. *Annual survey of statistical technique developments in the analysis of multivariate data*, Part I, Econometrica, vol. 4 (1936), pp. 264–268. See also C. F. Roos, *Annual survey of statistical techniques: The correlation and analysis of time series*, Part II, Econometrica, vol. 4 (1936), pp. 368–381.

averages or parameters calculated from a sample. In what follows, my apparent freedom will be exercised by choosing to emphasize the "Student" ratio, although commenting briefly on some of the other concepts mentioned above and on certain of their generalizations. This choice of emphasis is perhaps largely one of expediency as it seems simpler to discuss, before a general mathematical audience, the concept and sampling theory of the "Student" ratio than of any one of the other concepts whose exact sampling theory is finding many applications.

It seems convenient to divide the paper into two parts. In Part I it will be assumed that the random samples are drawn from normal parent populations, whereas in Part II we shall deal briefly with random samples drawn from certain specified non-normal populations.

PART I

RANDOM SAMPLES FROM NORMAL PARENT POPULATION

2. *Distribution Function of the Sum of Squares.* While it seems appropriate to cite "Student's" paper* of 1908 as marking the beginning of what is commonly regarded as "small or exact sampling theory" in applied statistics, it also seems to be appropriate and historically correct to direct attention to papers by Helmert† published in 1875–1876 that gave the basis for starting a new small sampling development from his theorems concerning the theoretical distribution function of the sums of squares of true and of apparent errors. In the language of statistics, these theorems may be expressed as follows.

Given a normal parent population of x's with mean 0 *and variance* σ^2 *from which are drawn at random each of N independent values,* $x_1, x_2, \cdots, x_N$, *measured from the population mean as the origin, giving as the sample mean* $\bar{x}=(x_1+x_2+\cdots+x_N)/N$ *and as the second moment of the sample from the population mean* $s^2=\bar{\mu}=(x_1^2+x_2^2+\cdots+x_N^2)/N$. *Then the probability that the*

* Loc. cit.

† *Ueber die Wahrscheinlichkeit der Potenzsummen der Beobachtungsfehler und über einige damit im Zusammenhange stehende Fragen,* Zeitschrift für Mathematik und Physik, vol. 21 (1876), pp. 192–218; see also Astronomische Nachrichten, vol. 85 (1875), No. 2039; ibid, vol. 88 (1876), No. 2096–7. See also E. Czuber, *Theorie der Beobachtungsfehler,* 1891, pp. 136–164.

sum of squares of deviations $U=x_1^2+x_2^2+\cdots+x_N^2$ *will fall into the interval* U *to* $U+dU$ *is given* by*

$$(1) \qquad \frac{1}{2^{N/2}\sigma^N\Gamma\left(\frac{N}{2}\right)} U^{(N-2)/2}e^{-U/(2\sigma^2)}dU.$$

If we let $U=N\bar{\mu}$, we obtain as a corollary of (1) that the probability that a sample value of $\bar{\mu}$ will fall into the interval $d\bar{\mu}$ at $\bar{\mu}$ is given by

$$(2) \qquad \left(\frac{N}{2}\right)^{N/2}\frac{1}{\sigma^N\Gamma\left(\frac{N}{2}\right)}\bar{\mu}^{(N-2)/2}e^{-N\bar{\mu}/(2\sigma^2)}d\bar{\mu}.$$

The distribution functions (coefficients of dU and $d\bar{\mu}$) in (1) and (2) were derived by Helmert in a rather elegant but somewhat tedious manner involving mathematical induction. In statistical applications we do not ordinarily know the mean of the population nor the variance, σ^2, but make estimates of their values from a random sample. To deal with this situation, Helmert found that the probability that the sum of squares of residuals

$$(3) \qquad U = N\bar{\mu} = (x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_N-\bar{x})^2$$

will fall into the interval dU at U is the same as the probability that the sum of squares of $N-1$ discrepancies from the population mean will fall into the same interval, so that the frequency function of the sample variance $s^2=\bar{\mu}$ given by (3) is equal to

$$(4) \qquad \left(\frac{N}{2}\right)^{(N-1)/2}\frac{1}{\sigma^{N-1}\Gamma\left(\frac{N-1}{2}\right)}\bar{\mu}^{(N-3)/2}e^{-N\bar{\mu}/(2\sigma^2)}.$$

The frequency functions (1), (2), and (4) are Pearson Type III functions. Although Czuber in his *Beobachtungsfehler* (1891) gave, in an improved notation, a good account of Helmert's contributions, neither the theoretical nor the practical import of the discovery was recognized in practical statistics until long

* It is to be understood throughout this paper that "is given by," used before an element for integration, means "is given, to within infinitesimals of higher order, by."

after Pearson in 1900 had arrived at the distribution of his χ^2 and "Student" had in 1908 inferred the distribution of the variances, s^2, from the relations he had found among the expected values of the first four moments of variance.

"Student" was first to recognize the fundamental importance, for the theory of small samples, of taking account of the simultaneous sampling variations of $\bar{x}$ and s since the ratio, $\bar{x}/s$, is used whenever we enter a normal probability table with s as an estimate of σ. On the basis of finding a linear correlation equal to zero between $\bar{x}^2$ and s^2, "Student" correctly inferred, by means of a remarkable intuition, the independence of $\bar{x}$ and s^2 in the probability sense of independence. This property of independence of $\bar{x}$ and s^2 was established explicitly by R. A. Fisher* by showing that the simultaneous distribution function of $\bar{x}$ and s^2 is the product of two functions, one of which contains $\bar{x}$ but not s and the other contains s but not $\bar{x}$, and it was implicitly contained in earlier work by Karl Pearson† and others.‡

By considering the ratio $z=\bar{x}/s$ and taking into account the sampling variations of s as well as those of $\bar{x}$, "Student" found the probability that a random sampling value of z will fall into an assigned dz to be

$$(5) \qquad \frac{\Gamma(N/2)dz}{\pi^{1/2}\Gamma\left(\frac{N-1}{2}\right)(1+z^2)^{N/2}}.$$

For values of $N>30$, it is frequently satisfactory to employ the normal function

$$\left(\frac{N-3}{2\pi}\right)^{1/2} e^{-(N-3)z^2/2}$$

for (5) in applications.

A small probability table of the integral of (5) for $N=4$ to $N=10$ is given by "Student"§ in his original paper and later in a somewhat more extensive table‖ for $N=2$ to $N=30$ in 1915.

* R. A. Fisher, *Applications of "Student's" distribution*, Metron, vol. 5 (1925), No. 3, pp. 90–93.

† Karl Pearson, *On the distribution of the standard deviations of small samples*, Biometrika, vol. 10 (1915), pp. 522–529.

‡ U. Romanowsky, *On the moments of standard deviations and of correlation in samples from a normal population*, Metron, vol. 5 (1925), No. 4, pp. 8–12.

§ Loc. cit., p. 19.

‖ Biometrika, vol. 11, pp. 414–417.

These tables give the probability (to four places of decimals) that a sample value of z shall fall between $-\infty$ and an assigned z.

The revolutionary character of the idea introduced by "Student" comes forcibly to light by making applications that involve drawing probable inferences from small samples, say from a sample with $N = 10$, and perhaps even more forcibly in following the generalizations of the "Student" idea.

3. *The Fisher Modifications and Generalizations.* For purposes of generalization, it seems that, in tests of significance, there are some advantages in following R. A. Fisher's modification of the "Student" ratio by using the ratio of $\bar{x}$ to its own standard deviation, $s/(N-1)^{1/2}$, as estimated from a sample. Then with

$$t = \frac{\bar{x}}{s}(N-1)^{1/2} \qquad \text{or} \qquad z = \frac{\bar{x}}{s} = \frac{t}{(N-1)^{1/2}},$$

the probability that t, from a sample of N, will fall into dt is given by

$$(6) \qquad \frac{\Gamma\left(\frac{N}{2}\right) dt}{(N-1)^{1/2}\pi^{1/2}\Gamma\left(\frac{N-1}{2}\right)\left(1+\frac{t^2}{N-1}\right)^{N/2}}.$$

The extensions of the "Student" ratio by R. A. Fisher may be described as schemes for making the "Student" idea applicable to a wide range of data largely by emphasis on building a variable t in the nature of a fraction whose numerator is a variate normally distributed about zero and whose denominator is an unbiassed estimate of the standard deviation of the numerator. Fisher applied this principle in the building of appropriate variables to obtain new tests of the significance of the difference between two means, of a linear regression coefficient, of non-linear regression coefficients of orthogonal functions, and of the coefficients in a multiple regression surface.

4. *Generalization of the "Student" Ratio.** Both applications to situations involving more than one variable and the natural

* Harold Hotelling, *The generalization of "Student's" ratio*, Annals of Mathematical Statistics, vol. 2 (1931), pp. 360–378.

tendency to generalize important ideas probably suggested the generalization of the "Student" ratio to multivariate situations. Hotelling's generalization sets up a function T of n variables $x_1, x_2, \cdots, x_n$, each of which is measured for N individuals.

To obtain a general notion of the nature of T, consider first the deviations of sample values from a hypothetical set of mean values $m_1, m_2, \cdots, m_n$. We may calculate the means $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$, of the samples, and put $\xi_i = (\bar{x}_i - m_i)/N^{1/2}$. Assuming the individual items to be taken independently from an infinite population, the expected value of ξ_i will be zero. To outline a procedure for estimating the values of the variance and covariances, we may write $x_{ik} = X_{ik} - \bar{x}_i$, where X_{ik} is the value of x_i for the kth individual. Then take

$$a_{ij} = a_{ji} = \frac{1}{N-1} \sum_{k=1}^{N} x_{ik} x_{jk}, \tag{7}$$

and

$$A_{ij} = A_{ji} = (\text{cofactor of } a_{ij} \text{ in } |a_{ij}|)/|a_{ij}|,$$

where $|a_{ij}|$ denotes the nth order determinant of elements a_{ij}. Then the measure of simultaneous deviations employed in the generalization is the quadratic form

$$T^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \xi_i \xi_j. \tag{8}$$

In problems of examining the deviation of a single variable from zero, T reduces to the Fisher modification of the "Student" ratio. For problems of examining the deviations from zero of two variables, say x and y, T^2 reduces to

$$T^2 = \frac{N}{1-r^2} \left(\frac{\bar{x}^2}{s_x^2} - \frac{2r\bar{x}\bar{y}}{s_x s_y} + \frac{\bar{y}^2}{s_y^2} \right)$$

in a familiar notation involving sample means, variances, and the coefficient of correlation.

Illustrations of the situations to which the T^2 may be applied include the comparison of means of two samples of n variables, and comparison of regression coefficients for functions of more than one variable. Deviations, x_{ik}, of the observations from means, or from regression functions, or other such functions, are

used to estimate the variances and covariances, a_{ij}. The distribution function of T is found to be given by

$$(9) \qquad \frac{2\Gamma\left(\frac{N}{2}\right)T^{n-1}}{\Gamma\left(\frac{N-n}{2}\right)\Gamma\left(\frac{n}{2}\right)(N-1)^{n/2}\left(1+\frac{T^2}{N-1}\right)^{N/2}}.$$

5. *Generalization of Variance.* In 1932, Wilks* gave generalizations to n-variate populations of the concept of variance and found the distributions of several important functions of the sample observations whose distribution laws had been established when the sampling was limited to univariate normal populations.†

More specifically, the extensions in question are concerned with the generalization of the concept of variance itself, and of such other concepts as the Fisher z, the correlation ratio, γ, the related $1-\gamma^2$, the "Student" ratio previously generalized by Hotelling, the λ-criteria of Pearson and Neyman, so as to give each of these concepts a meaning for multivariate populations and for samples from such populations. In his article of 1934 giving "recent advances," Irwin states very appropriately that this paper by Wilks gives generalizations of the greatest interest.

In a notation differing only slightly from that used above, Wilks takes

$$(10) \qquad a_{ij} = \frac{1}{N}(X_{ik} - x_i)(X_{jk} - x_j), \qquad (i, j = 1, 2, \cdots, n).$$

The determinant $|a_{ij}|$ was adopted as the generalized sample variance. Wilks points out that $|a_{ij}|$ for n-variate samples and the ordinary variance for univariate samples are similar in the way they arise in maximizing likelihood functions.

6. *Distribution of Generalized Variance.* Wilks found the distribution function of $|a_{ij}|$ by the method of moment generating

* S. S. Wilks, *Certain generalizations in the analysis of variance*, Biometrika, vol. 24 (1932), pp. 471–494; *Moment-generating operators for determinant of product moments in samples from a normal system*, Annals of Mathematics, (2), vol. 35 (1934), pp. 312–339.

† See J. Wishart, *The generalized product moment distribution in samples from a normal multivariate population*, Biometrika, vol. 20A (1928), pp. 32–52.

functions. His success seems to depend largely on the fact that he obtained the solutions of two integral equations that have a wide application. The distribution function of $\xi = |a_{ij}|$ took the form of a multiple integral which could be integrated and expressed explicitly for special values of the parameters and for $n=1$ and $n=2$. Thus, for $n=1$, the distribution function reduced to the well known distribution of ordinary variance given in (4). For $n=2$, the distribution function for generalized variance is

$$D_2(\xi) = \frac{2^{N-3}A_2^{(N-2)/2}\xi^{(N-4)/2}e^{-2(A_2\xi)^{1/2}}}{\Gamma(N-2)}, \tag{11}$$

where

$$A_2 = \frac{N^2}{4\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}.$$

In 1934, Kullback* obtained the distribution function of ξ in explicit form for any positive integral value of n, by the use of characteristic functions.

7. *Distribution of the Ratio of Generalized Variances.* Fisher's z-distribution in which $z=\frac{1}{2}\log s_2^2/s_1^2$, where s_1^2, and s_2^2 are two independent estimates of variance, is much used in the analysis of variance for univariate populations. Put $u=s_2^2/s_1^2=e^{2z}$. It is this u which Wilks has generalized into $\psi=\xi/\eta$, where ξ and η are generalized variances for two samples from populations of n variables for which the generalized variances are given. He then developed the distribution function of ψ, and, in particular, the distribution function of ψ for the bivariate population as a special case.

8. *Generalized Correlation Ratio.* Consider p samples ω_β, $(\beta=1, 2, \cdots, p)$, of N_β items respectively drawn from a normal population of one variable. Let $\bar{x}_\beta$ and s_β^2 be the mean and variance of ω_β. Let Ω be the sample formed by pooling the ω's, and let its mean and variance be denoted by $\bar{X}$ and S^2, respectively. Then the correlation ratio, η, for the p categories, as defined by R. A. Fisher, is given by

* Solomon Kullback, *An application of characteristic functions to the distribution problem of statistics*, Annals of Mathematical Statistics, vol. 5 (1934), pp. 263–307.

$$\eta^2 = \frac{\sum_{\beta=1}^{p} N_\beta(\bar{x}_\beta - \bar{x})^2}{NS^2}, \qquad \left(\sum_{\beta=1}^{p} N_\beta = N\right), \tag{12}$$

this being a weighted variance of the means of p sub-samples (arrays) divided by the variance of Ω. To generalize (12), consider the p samples ω'_β, $(\beta = 1, 2, \cdots, p)$, of n_β items drawn from an n-variate normal population. Let the sample formed by pooling the drawings be Ω' with $\sum n_\beta = N$ items. Then the generalized η^2 may be written

$$U = \frac{|b_{ji}|}{|a_{ij}|}, \tag{13}$$

where

$$b_{ij} = b_{ji} = \frac{1}{N}\sum_{\beta=1}^{p} n_\beta(\bar{x}_{i\beta} - \overline{X}_i)(\bar{x}_{j\beta} - \overline{X}_j),$$

and

$$a_{ij} = a_{ji} = \frac{1}{N}\sum_{\beta=1}^{p}\sum_{k=1}^{n_\beta}(x_{i\beta k} - \overline{X}_i)(x_{j\beta k} - \overline{X}_j),$$

where $\overline{X}_{i\beta}$ is the mean of the ith variate in the βth sample and $x_{i\beta k}$ is the value of the kth individual of the ith variate in the βth sample, ω'_β.

The distribution function of U is found. The integrations are carried out for $n = 1$ and yield the well known results of R. A. Fisher and of Hotelling for the distribution of η^2. We may write $a_{ij} = b_{ij} + c_{ij}$, where

$$c_{ij} = \frac{1}{N}\sum_{\beta=1}^{p}\sum_{k=1}^{n_\beta}(x_{i\beta k} - \overline{X}_{i\beta})(x_{j\beta k} - \overline{X}_{j\beta}).$$

It is shown that $W = |c_{ij}| / |a_{ij}|$ may be regarded as a generalization of $1 - \eta^2$, and the distribution function of W is found. For $n = 1$, the integrations are carried out completely. Moreover, it is shown that W serves as a maximum likelihood criterion, λ_H, of the type used by Neyman and E. S. Pearson. It turns out that a simple relation $\lambda_H = W^{n/2}$ exists, thus showing a simple connection among these fundamental statistical concepts.

For samples from multivariate populations, Wilks* has devised criteria of the Neyman-Pearson likelihood type for testing the following classes of hypotheses:

(1) That a sample is from a normal population with a specified set of means.

(2) That two or more samples are from populations having a common system of: (a) means; (b) variances and covariances; (c) means, variances, and covariances.

(3) That several sets of variates are mutually independent.

Wilks directs attention to the fact that all the criteria he has considered may be called "Studentized" functions by which he means that the criteria and their probability functions are completely expressible in terms of observations.

PART II

RANDOM SAMPLES FROM SOME NON-NORMAL PARENT POPULATIONS

9. *Non-Normal Parent Populations.* It is fairly obvious that many of the samples used in the application of statistical coefficients or ratios are drawn from non-normal parent populations that differ very much from normal populations. Moreover, a small sample is almost sure to be inadequate to give information essential to pass a reasonable judgment about the type of parent population. This situation suggests the importance of learning, if possible, how far departures from normality in a parent population influence the distribution of statistical estimates such as the mean, the variance, and the "Student" ratio used to decide whether a sample belongs to a given parent population.

We shall consider first a few non-normal parent populations for which the exact form of the distribution function of the mean, standard deviation, or "Student" ratio has been found. Incidentally, in some cases, we shall comment also very briefly on the exact distributions of some other averages such as the median, harmonic mean, geometric mean, center, or variance, if time and space permit. Later, we shall discuss briefly the progress that has been made in characterizing the distributions of $\bar{x}$, s^2, and $\bar{x}/s$ for samples from various types of parent populations by means of theoretical moments and by experimental sampling.

* S. S. Wilks, *Test criteria for statistical hypotheses involving several variables*, Journal of the American Statistical Association, vol. 30 (1935), pp. 549–560.

10. *On Exact Distributions of Some Averages for Samples from Certain Non-Normal Parent Populations with Special Reference to the Mean* $\bar{x}$. For a continuous rectangular parent population given by $y=1/a$ from $x=0$ to $x=a$, $(a>0)$, the distribution function of $\bar{x}$ for samples of N items consists of N polynomials, each being of degree $N-1$, and each being applicable to a subinterval of length a/N. The curve is bell-shaped and resembles the normal curve when $N\geqq 3$. For $N=2$, the curve degenerates into the two equal sides of an isosceles triangle.

It seems nearly certain that Laplace* knew the distribution of means of samples of N items, each drawn from a continuous rectangular universe; for he knew the exact frequency function, $f(u)$, of the sum $u=x_1+x_2+\cdots+x_N$ of N elements, each x_i being a real number taken at random from a given interval 0 to a, $(a>0)$, of a rectangular universe, and it is a small step from this distribution of u to the distribution of the corresponding mean $\bar{x}$, where $\bar{x}=Nu$. Laplace applied the distribution function of u to the solution of the historic problem of the probability that the inclinations of the orbits of the ten planets besides the earth known at the beginning of the year 1801 do not constitute a random distribution. Perhaps it would not be without some interest to remark that the distribution formula for the sum, u, of N elements drawn at random from a rectangular universe as described above, has again appeared in 1936 in an up-to-date problem† in a paper by Condon and Breit, on the energy distribution of neutrons. The distribution of means of samples drawn from a continuous rectangular universe was given in explicit form by Irwin‡ and by Hall§ in 1927.

In 1929, Rider‖ developed the exact distribution function of

* See H. L. Rietz, *On a certain law of probability of Laplace*, Proceedings, International Congress of Mathematics, Toronto, vol. 2 (1924), pp. 795–799.

† E. U. Condon and G. Breit, *The energy distribution of neutrons slowed by elastic impacts*, Physical Review, (2), vol. 49 (1936), No. 3, p. 230.

‡ J. O. Irwin, *On the frequency distribution of the means of samples from a population having any law of frequency with finite moments, with special reference to Pearson's Type* II, Biometrika, vol. 19 (1927), pp. 225–239.

§ Phillip Hall, *The distribution of means of samples of size N drawn from a population in which the variate takes values between* 0 *and* 1, *all such values being equally probable*, Biometrika, vol. 19 (1927), pp. 240–244.

‖ P. R. Rider, *On the distribution of ratio of mean to standard deviation in small samples from non-normal universes*, Biometrika, vol. 21 (1929), pp. 124–143.

means of samples of four items drawn from a discrete five class rectangular parent population with x taking values from -2.5 to $+2.5$. He found that third differences of the resulting probabilities vanish except at $x=0, \pm 0.25$. He fitted the probabilities by means of cubic curves, and also exhibited the results in tabular form. Similar results were obtained by using curves of degree one and two for samples of two and three items, respectively.

The paper to which I have just now referred presents some exact distributions of medians and of some other averages for both discrete or continuous rectangular parent populations. However, it is concerned largely with exact distributions of "Student" ratios to be discussed later in this paper. E. L. Dodd* had given a formula in 1922, in terms of integrals, for the distribution of medians of samples from a rather general parent population, but Rider gave the results in explicit integrated form for the rectangular parent population.

In 1920, Karl Mayr† developed a theory for the determination of the distribution of the sum of N items. He then applied the theory to the parent population given by $\phi(x)=e^{|-x|}/2$ for all real values of x.

The determination of the distribution functions of sample means for the Pearson types as parent populations has been the subject of considerable investigation by several authors. A. E. R. Church‡ seems to have been first to find the exact distribution of the mean, $\bar{x}$, of samples from the Type III population. He carried out the integrations. The resulting distribution function is a Type III curve. J. O. Irwin§ arrived a little later at the same result by a method that involves the use of integral equations and complex variables. He found also, in the form of an integral, the distribution of the means of samples from a Type II population, and evaluated the integrals in some special

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* E. L. Dodd, *Functions of measurements under general laws of error*, Skandinavisk Aktuarietidskrift, vol. 5 (1922), pp. 132–158.

† Karl Mayr, *Wahrscheinlichkeitsfunctionen und ihre Anwendungen*, Monatshefte für Mathematik und Physik, vol. 30 (1920), pp. 17–43.

‡ A. E. R. Church, *Means and squared standard deviations of small samples from any population*, Biometrika, vol. 18 (1926), pp. 321–394.

§ Loc. cit., pp. 225–239.

cases. In 1929, C. C. Craig* obtained the distribution of sample means from a Type III parent population by the use of semi-invariants of Thiele. In 1930, G. A. Baker† found the distribution function of means, $\bar{x}$, of samples of N drawn from a parent population defined by the first $m-1$ terms of a Type A Gram-Charlier series

$$f(x) = a_0\phi_0(x) + a_3\phi_3(x) + \cdots + a_m\phi_m(x),$$

where

$$\phi_i(x) = \frac{d^i(e^{-x^2/2})}{dx^i}.$$

In 1931, C. C. Craig‡ arrived at Baker's results in a very few steps, by use of the semi-invariants of Thiele. In 1931, Rider§ gave the distributions, in tabular form, for means of samples drawn from triangular and from U-shaped populations.

In 1932, A. T. Craig‖ classified probability functions, $f(x)$, of the parent populations from which samples are drawn, into three classes according as x is allowed the range $(-\infty, \infty)$, $(0, \infty)$, or $(0, A)$, $A>0$. Craig then established general theorems by which the problems of finding the distributions of arithmetic means, harmonic means, geometric means, medians, quartiles, and deciles are reduced to problems of integration. The illustrative parent universes for which integrations were carried out for the arithmetic mean, for some or all values of N, are given

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* C. C. Craig, *Sampling when the parent population is of Pearson's Type* III, Biometrika, vol. 20 (1929), pp. 287–293.

† G. A. Baker, *Distribution of the means of samples of n drawn at random from a population represented by a Gram-Charlier series*, Annals of Mathematical Statistics, vol. 1 (1930), pp. 199–204. See also B. H. Camp, *Problems in sampling*, Journal of the American Statistical Association, vol. 18 (1923), pp. 964–977.

‡ C. C. Craig, *Note on the distribution of means of samples of N drawn from a Type A population*, The Annals of Mathematical Statistics, vol. 2 (1931), pp. 99–101.

§ P. R. Rider, *On small samples from certain non-normal universes*, Annals of Mathematical Statistics, Vol. 2 (1931), pp. 48–65.

‖ A. T. Craig, *On the distribution of certain statistics*, American Journal of Mathematics, vol. 54 (1932), pp. 353–366.

by assigning to the parent population, $f(x)$, the following special forms:

$$(1) \qquad f(x) = \frac{1}{a}, \qquad (0 \leqq x \leqq a),$$

$$(2) \qquad f(x) = \frac{k}{\sigma} e^{-x/\sigma}, \qquad (0 \leqq x < \infty),$$

$$(3) \qquad f(x) = kx^{-1/2}e^{-x/2}, \qquad (0 \leqq x < \infty),$$

$$(4) \qquad f(x) = \frac{k}{\sigma} e^{-|x|/2}, \qquad (-\infty < x < \infty),$$

$$(5) \qquad f(x) = \frac{4x}{a^2}, \qquad (0 \leqq x \leqq a/2),$$

$$= \frac{4}{a^2}(a - x), \qquad (a/2 \leqq x \leqq a).$$

The integration for (4) is carried out for $N=2$, 3, and 4 only, and that for (5) is carried out for $N=2$ only. For the distributions of harmonic means, geometric means, medians, and ranges, a somewhat similar set of illustrations is given for which integrations are carried out.

To illustrate, for the median, the integrations were carried out for the following besides the rectangular distribution:

$$f(x) = \frac{k}{\sigma} e^{-|x|/\sigma}, \qquad (-\infty < x < \infty),$$

$$f(x) = \frac{2x}{a^2}, \qquad (0 \leqq x \leqq a),$$

$$f(x) = e^{-x}, \qquad (0 \leqq x < \infty).$$

In 1934, Baten* found in explicit form, the distribution function for the sum of N independent items of a sample from a parent population defined by $(1/(2h))$ sech $(\pi x/(2h))$.

In a paper of 1935,† Weida developed the distribution func-

* W. D. Baten, *The probability law for the sum of n independent variables, each subject to the law* $(1/(2h))$ sech $(\pi x/(2h))$, this Bulletin, vol. 40 (1934), pp. 284–290.

† F. M. Weida, *On certain distribution functions when the law of the universe is Poissons' First Law of Error*, Annals of Mathematical Statistics, vol. 6 (1935), pp. 102–110.

tion for the samples of N items for the parent population

$$f(x) = \frac{k}{\sigma} e^{-|x|/\sigma}, \qquad (-\infty < x < \infty),$$

in explicit form, for any value of N. The development was carried out by the use of characteristic functions.

11. *On Some Exact Distributions of Standard Deviations.* With respect to exact distributions of sample standard deviations, s, from non-normal parent populations, Rider* found, for samples of 2 items, the distribution function of s to be $f(s) = 4(1-2s)$, when the drawings are from the rectangular parent universe given by $y = 1$ from $x = 0$ to $x = 1$.

Rietz† found the exact distribution function of the standard deviation of samples of 3 drawn from a rectangular parent population given by $y = 1/a$ from $x = 0$ to $x = a$ to be

$$f(s) = \frac{6 \cdot 3^{1/2}}{a^2} (\pi a - 3s6^{1/2})s \qquad \text{when} \qquad 0 \leqq s \leqq \frac{a}{6} 6^{1/2},$$

$$f(s) = \frac{18}{a^2}\left[- a3^{1/2} \arccos\left(\frac{a^2}{3s^2} - 1\right) + 2(18s^2 - 3a^2)^{1/2} + \frac{\pi}{3} a3^{1/2} - 3s2^{1/2}\right] \qquad \text{when} \qquad \frac{a}{6} 6^{1/2} \leqq s \leqq \frac{a}{3} 2^{1/2},$$

where s is the standard deviation of samples.

It is easily shown that there is continuity both in the two curves and in their slopes at the junction at $s = a6^{1/2}/6$. We may note that the distribution curve for the interval 0 to $a6^{1/2}/6$ on s is a parabola, but the distribution curve for the interval $a6^{1/2}/6$ to $a2^{1/2}/3$ on s is a transcendental curve with a rather complicated equation. When $a = 4$, we find 98.86 per cent of the area under the curve is under the parabola to the left of the ordinate $s = a6^{1/2}/6$ at the junction with the more complicated curve.

Rider‡ directed attention to the fact that several of the es-

* Loc. cit., p. 141.

† H. L. Rietz, *Note on the distribution of the standard deviation of sets of three varieties drawn at random from a rectangular population*, Biometrika, vol. 23 (1931), pp. 424–426.

‡ Loc. cit. (1929), p. 139.

timates of parameters (mean, median, range, and extreme average) in samples from a rectangular universe are distributed in accord with polynomials, which are, apparently, of degree one less than the number in the sample. This rule still holds for the distribution of the standard deviation from two items and for the major portion of the range for the case of three items. It seems natural to surmise that the distribution of s from N items may be a polynomial of degree $N-1$ for a part of the range on s starting at zero. My attempts to prove or disprove this conjecture have been unsuccessful. However, I shall present a bit of experimental evidence that the distribution of s for samples of four items drawn from a rectangular population is a polynomial of degree 3 for an interval at the left end of the range. Using Tippett's Random Numbers, we* drew 400 samples of 4 items each from a rectangular distribution with 19 class intervals, and obtained the following distribution of standard deviation s:

s	Frequency	s	Frequency
0.000 – 0.249	1	2.500 – 2.749	38
0.250 – 0.499	5	2.750 – 2.999	42
0.500 – 0.749	5	3.000 – 3.249	38
0.750 – 0.999	15	3.250 – 3.499	20
1.000 – 1.249	24	3.500 – 3.749	11
1.250 – 1.499	24	3.750 – 3.999	3
1.500 – 1.749	28	4.000 – 4.249	2
1.750 – 1.999	44	total	400
2.000 – 2.249	49		
2.250 – 2.499	51		

Without taking the space to discuss our scheme of testing this distribution, I will merely state that it turns out, when degrees of freedom are taken into account, that a third degree polynomial will fit the first 12 class frequencies of the 17 shown above better than a fourth or fifth degree polynomial. This conforms to Rider's observation.

* A. C. Olshen made the drawings.

In 1935, G. A. Baker* found the distribution function of the standard deviation for samples of 2 drawn from a parent population given by the first three terms of the Gram-Charlier Type A, and the distribution function for the standard deviation of samples of 3 drawn from a parent distribution given by the first two terms of the Type A series.

In 1932, A. T. Craig† found in the form of integrals the simultaneous frequency functions of $\bar{x}$ and s for drawings from a general type of non-normal parent populations when $N=2$, 3, or 4. The integrations were carried out in explicit form for several special functions when $N=2$, and $N=3$.

12. *On Some Exact Distributions of the "Student" Ratio.* In his 1929 paper,‡ Rider gave, in tabular form, the distributions of "Student" ratios for samples of 2, 3, and 4 items drawn from discrete rectangular parent populations of five classes, and also of ten classes for samples of 4. Some features of the distributions will be discussed later in this paper. Rider§ gave also the distribution function of the "Student" ratio for samples of two items drawn from a continuous rectangular parent population. In 1931, he published‖ the results of his study of samples from both triangular and U-shaped distributions. He drew the inferences that the general characteristics of the z-distribution from the U-shaped parent population resemble those for the rectangular population, that the negative skewness in the triangular population tended to produce skewness of the opposite type¶ in the distribution of the "Student" z, and that the cumulative probability of $|z|$, for the triangular population, tends to follow the results from a normal universe fairly well.

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* G. A. Baker, *Note on the distributions of the standard deviations and second moments of samples from a Gram-Charlier population*, Annals of Mathematical Statistics, vol. 6 (1935), pp. 127–130.

† A. T. Craig, *The simultaneous distribution of mean and standard deviation in small samples*, Annals of Mathematical Statistics, vol. 3 (1932), pp. 126–140.

‡ Loc. cit., pp. 124–143.

§ An error in sign has been corrected by Victor Perlo. Reference cited later.

‖ P. R. Rider, *On small samples from certain non-normal universes*, Annals of Mathematical Statistics, vol. 2 (1931), pp. 48–65.

¶ See Neyman and E. S. Pearson, loc. cit., Biometrika, vol. 20A (1928), p. 198. See also "Sophister", loc. cit., Biometrika, vol. 20A (1928), p. 408.

In 1933, Victor Perlo* found the distribution functions of the "Student ratio" as modified by R. A. Fisher, for samples of 2 items, drawn from a rectangular parent population, to be

$$\frac{1}{2(1+|t|)^2};$$

for samples of 3, to be

(i)
$$\frac{-9}{4(t+1)(t^2-4)}\left(\frac{1}{t+1}+\frac{3t}{t^2-4}\right) + \frac{3^{3/2}(t^2+2)}{(t^2-4)^{5/2}}\tan^{-1}\frac{(t^2-4)^{1/2}}{3^{1/2}(t+2)}, \qquad (\infty \geqq t \geqq 2),$$

(ii)
$$\frac{-9}{4(t+1)(t^2-4)}\left(\frac{1}{t+1}+\frac{3t}{t^2-4}\right) + \frac{3^{3/2}(t^2+2)}{(4-t^2)^{5/2}}\tanh^{-1}\frac{(4-t^2)^{1/2}}{3^{1/2}(t+2)}, \qquad (2 \geqq t \geqq \tfrac{1}{2}),$$

(iii)
$$\frac{3^{1/2}}{2(4-t^2)(1-t^2)^{1/2}}\left(1-\frac{9t^2}{4-t^2}\right) + \frac{3^{3/2}(t^2+2)}{(4-t^2)^{5/2}}\tanh^{-1}\left(\frac{1-t^2}{4-t^2}\right)^{1/2}, \qquad (\tfrac{1}{2} \geqq t \geqq 0).$$

The function (i), (ii), (iii), is continuous with continuous derivatives except at $t=\pm 1/2$. For these values of t, the derivative is discontinuous. When this distribution function is compared with "Student's" exact distribution for samples of 3 from a normal parent population, the frequency of values of the "Student" ratio is greater at the ends and the middle, and less elsewhere, when the parent distribution is rectangular than when it is normal.

In bringing to a conclusion our comments on exact distribution functions, perhaps it should be said that it is not lack of interest, but only limitations of time and space that cause me to make my comments so brief on the contributions concerned with the distribution of averages other than the mean and standard deviation. No comments have been made on some

* Victor Perlo, *On the distribution of Student's ratio for samples of three drawn from a rectangular distribution*, Biometrika, vol. 25 (1933), pp. 203–204.

very interesting cases. Among these are the exact explicit distribution functions which Neyman* and E. S. Pearson found, from a rectangular parent population, for the distribution of range, center, and ratio of deviation of sample center from population center to the semirange.

13. *Investigations of the Distributions of $\bar{x}$ and s^2 by Moments and by Experimental Sampling.* In 1928, Church† contributed some experimental sampling investigations that are especially important when viewed as an attempt to make use of theoretical moments of mean and variance to obtain Pearson frequency curves to serve as distribution functions of the mean and variance. He found the distribution of the means of samples of 10 drawn from two infinite skew populations and from one finite population. In his interpretation of the results, he emphasized the strong tendency of the distribution of the means, $\bar{x}$, to assume an approximately normal form, and gave a method of predicting rather rapidly the Pearson type to which the distribution of $\bar{x}$ is likely to approximate in the case of samples drawn from an infinite supply. Later, in a study of means of samples from a *U*-shaped parent population, Holzinger‡ and Church found that the distribution of means, $\bar{x}$, in samples of N obtained by sampling from a *U*-shaped population is quite unsatisfactorily represented by a simple continuous curve until N is at least of the order of 50. It is further inferred that this effect is due mainly to the fact that when N is quite small, the distribution of $\bar{x}$ is likely to be composite in form. Returning to the part of Church's work on sample variance, s^2, we observe that it was inferred that the distribution of s^2 from the samples of 10, from the infinite supply, for each of the two populations may be described by Pearson curves in a manner useful for applied statistics.

"Sophister"§ extended the experimental work of Church by

* Neyman and E. S. Pearson, *On the use and interpretation of certain test criteria for purposes of statistical inference*, Biometrika, vol. 20A (1928), pp. 175–240.

† Loc. cit., pp. 321–394.

‡ K. J. Holzinger and A. E. R. Church, *On the means of samples from a U-shaped population*, Biometrika, vol. 20A (1928), pp. 361–388.

§ "Sophister", *Discussion of small samples drawn from an infinite skew population*, Biometrika, vol. 20A (1928), pp. 389–423.

giving the distribution of variance for samples of $N=5$ and $N=20$ items drawn from a Pearson Type III population differing much from a normal population. He inferred that the distribution of variances was adequately described by a Type VI curve. Attention may be called to the fact that Karl Pearson devised an equation for the distribution function of variances by assuming a Type VI curve which starts at zero and has its origin at zero, thereby using the constants of the population sampled only up to β_4.

In 1931 Le Roux* gave a numerical and graphic analysis of the formulas for the moments of variance with special reference to the relations among the β's with a view to obtaining suitable Pearson curves to represent the sampling distributions of statistics in the case of parent populations which can themselves be represented by Pearson curves. By this analysis light was thrown on the manner in which the variance distribution changes as the character of the population and sample size change. The methods were tested on twenty-one experimental distributions of s^2, among which are included "Sophister's" samples of 5 to 10 items, and Church's samples of 10 for one population. The inference is drawn that Karl Pearson's fixed start method of fitting the distribution of s^2 is satisfactory, giving for "goodness of fit" tests an average of $P=0.49$, whereas the four moment method of fitting may be quite unsatisfactory or even impossible in cases of small samples.

14. *More About the "Student" Ratio.* An investigation of the "Student" z from samples drawn from the skew populations used by "Sophister",† showed that the distributions of z were markedly skew. Nevertheless, he concluded that the value of "Student's" Table in practice is still indicated, even when the parent population is definitely skew.

It has been clearly shown by Shewhart that the theory of small samples is useful in applications to the quality control of manufactured articles. Shewhart‡ and Winters found, largely

* J. M. Le Roux, *A study of the distribution of variance in small samples*, Biometrika, vol. 23 (1931), pp. 134–190.

† Loc. cit., p. 408.

‡ W. A. Shewhart and F. W. Winters, *Small samples—new experimental results*, Journal of the American Statistical Association, vol. 23 (1928), pp. 144–153.

on the basis of experimental sampling, that the "Student" theory gives a marked improvement, for small samples, over the classical theory, but they indicated further that the "Student" procedure fails, in certain practical cases, to give a set of errors in means consistent with the actual errors they obtained from small samples. For the results in cases of sampling from both rectangular and triangular populations, part of the effect is traced to the correlation between $\bar{x}$ and s^2, and part to the departure of the distribution of s^2 from what it would be if the parent distribution were normal. Briefly stated, the probability that a value of z from observation will fall outside a prescribed interval from $-z_1$ to $+z_1$ is likely to be larger than the estimate from "Student's" probability tables, if the numerical value of z_1 is of moderate size, say in the neighborhood of 2 or more.

In 1929, Rider* gave further explanation for the failure of the "Student" probability tables to yield probabilities that would check well with the Shewhart and Winters experiments. After studying several types of parent populations, but especially the rectangular type, he also attributed the failure largely to the correlation between $\bar{x}$ and s^2 in the cases of non-normal parent distributions. In general, his results showed not only a greater number of numerical values of z outside an assigned interval $-z_1$ to z_1, as in the experiments of Shewhart and Winters, but also a greater clustering of numerical values of z about the population mean† than in the case of a normal parent population.

In 1929, E. S. Pearson‡ threw further light on the sensitiveness of values of the "Student" z so far as they concern the fundamental tests dealing with the means of samples, and the differences between means of two samples. He investigated the question as to how well the observed distributions of z, from a variety of non-normal populations, follow the "Student" theory based on a normal parent population. For a set of non-normal parent populations, the following classes of data were selected with $0 \leqq \beta_1 \leqq .5$:

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* Loc. cit., pp. 124–143.

† See Victor Perlo, loc. cit.

‡ E. S. Pearson, *The distribution of frequency constants in small samples from non-normal symmetrical and skew populations*, Biometrika, vol. 21 (1929), pp. 259–286.

1000 samples of 2 from 7 populations with $1.8 \leqq \beta_2 \leqq 7.07$,
500 samples of 5 from 6 populations with $2.5 \leqq \beta_2 \leqq 7.07$,
500 samples of 10 from 7 populations with $2.5 \leqq \beta_2 \leqq 7.07$,
500 samples of 20 from 6 populations with $2.5 \leqq \beta_2 \leqq 7.07$.

The inference is drawn that a completely satisfactory analysis of the position of the "Student" z-test will only be possible when the theoretical distribution of z in samples from the non-normal parent population in question is found. It seems to the writer that the probable date of finding such exact distributions, for rather general skew parent distributions, is far in the future. However, the experimental results reported in E. S. Pearson's excellent paper and elsewhere suggest some inferences as to the nature of departures of the distribution of z from that of the normal theory with sampling from a fairly wide range of populations. The least satisfactory agreement seems to occur in the cases of extremely leptokurtic parent distributions.

In concluding these remarks on the distribution of certain averages and ratios, for samples from non-normal parent populations, it seems fairly obvious that it is easy to propose a simple sounding problem by merely asking for the distribution function of some statistic for samples drawn from a simple parent population, but it is usually very difficult to solve the problem. On this account, we know relatively little at present of all we wish to know about the exact distributions of statistics for samples drawn from non-normal parent populations.

Although the prospects of obtaining the exact distribution functions of such statistics as the standard deviation, s, or the "Student" ratio, z, for samples from a considerable variety of non-normal populations, do not seem promising, nevertheless, by the use of moments of moments, and experimental sampling, along with the exact determination of some distribution functions, significant contributions are being made towards an understanding of the probable nature of certain important features of the distributions in question.

THE UNIVERSITY OF IOWA

A GENERALIZATION OF SCHWARZ'S LEMMA*

BY CONSTANTIN CARATHÉODORY

1. *Introduction.* We consider the family of functions $f(z)$, which are regular inside of the unit circle, which vanish at the origin, and whose absolute value $|f(z)|$ is less than one in that circle. Taking two points z_1 and z_2 in the interior of the unit circle we inquire about the maximum $M(z_1, z_2)$ of the expression

$$\left|\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right| \tag{1}$$

if $f(z)$ describes the family of functions considered above.

This maximum can never be less than one, because the function $f(z) \equiv z$ itself is contained among the functions of our family. But in a great number of cases $M(z_1, z_2)$ is *exactly equal to one.* Thus if z_1 is taken equal to zero, the assertion that $M(0, z_2) = 1$ is only another way of formulating the lemma of Schwarz. Again, if we assume that the ratio z_2/z_1 is real and negative, we have

$$|f(z_2) - f(z_1)| \leqq |f(z_1)| + |f(z_2)|,$$
$$|z_2 - z_1| = |z_1| + |z_2|;$$

and, using the lemma of Schwarz, we find that $M(z_1, z_2) = 1$.

In the third place, we have $M(z_1, z_2) = 1$ if *both* points z_1 and z_2 lie on the circular disc $|z| \leqq 2^{1/2} - 1$. This is an easy consequence of the fact that for all points of this figure the expression $|f'(z)|$ is never greater than one.† We are going to analyze the questions which arise from these different examples by determining completely all the cases for which $M(z_1, z_2) = 1$.

2. *An Auxiliary Function.* We begin with the obvious remark that our result will not be altered if we neglect from the outset all the functions of the form $f(z) = e^{i\theta}z$ for which the ex-

* From an address delivered before the Society under the title *Bounded analytic functions*, on November 27, 1936, by invitation of the Program Committee.

† J. Dieudonné, *Recherches sur quelques problèmes relatifs aux polynomes et aux fonctions bornées d'une variable complexe*, Annales de l'École Normale, (3), vol. 48 (1931), pp. 247–358; in particular, p. 352.

pression (1) is identically equal to one. For the remaining functions of our family we can put

$$f(z) = zg(z), \qquad \text{with} \qquad |g(z)| < 1, \tag{2}$$

and we may introduce the notation

$$g(z_1) = a. \tag{3}$$

The function $\phi(z)$ which is defined by the equation

$$\frac{a - g(z)}{1 - \bar{a}g(z)} = \frac{z_1 - z}{1 - \bar{z}_1 z}\phi(z), \tag{4}$$

is regular inside the unit circle; and it is readily seen that, for all points of that circle,

$$|\phi(z)| \leqq 1. \tag{5}$$

Using these formulas, we find

$$f(z) = z\frac{a(1 - \bar{z}_1 z) - (z_1 - z)\phi(z)}{(1 - \bar{z}_1 z) - \bar{a}(z_1 - z)\phi(z)}, \tag{6}$$

$$f(z_1) = z_1 a, \tag{7}$$

and finally

$$\frac{f(z_2) - f(z_1)}{z_2 - z_1} = \frac{a(1 - \bar{z}_1 z_2) + (z_2 - a\bar{z}_1)\phi(z_2)}{(1 - \bar{z}_1 z_2) - \bar{a}(z_1 - z_2)\phi(z_2)}. \tag{8}$$

Now the function (6) always belongs to our family provided $|a|$ be taken less than one and $\phi(z)$ be a regular analytic function inside the unit circle which satisfies the condition (5). Consequently the value of $M(z_1, z_2)$ can be obtained by calculating the maximum of the absolute value of the right-hand side of (8) under the conditions

$$|a| < 1, \qquad |\phi(z_2)| \leqq 1. \tag{9}$$

For a given value of a, the maximum value of this last expression is attained at some point of the unit circle $|\phi| = 1$, say at the point $-e^{i\alpha}$. But if we multiply both a and $\phi(z_2)$ by $e^{-i\alpha}$, the absolute value of the right-hand side of (8) remains unaltered. Hence we may also calculate $M(z_1, z_2)$ as the maximum of $|\omega(a)|$ for $|a| < 1$, if we define $\omega(a)$ by the relation

$$\omega(a) = \frac{a(1 - \bar{z}_1 z_2) - (z_2 - a\bar{a}z_1)}{(1 - \bar{z}_1 z_2) + \bar{a}(z_1 - z_2)}, \tag{10}$$

which we obtain from (8) by putting $\phi(z_2) = -1$.

3. *A Necessary and Sufficient Condition.* We remark that we may write

$$\omega(a) = \frac{a(1 - \bar{z}_1 z_2 + \bar{a}z_1) - z_2}{(1 - \bar{z}_1 z_2 + \bar{a}z_1) - \bar{a}z_2}, \tag{11}$$

and that, with the notation

$$u = \frac{z_2}{1 - \bar{z}_1 z_2 + \bar{a}z_1}, \tag{12}$$

ω takes the form

$$\omega = \frac{a - u}{1 - \bar{a}u}. \tag{13}$$

The absolute value of this last expression is always greater than one for $|u| > 1$, equal to one for $|u| = 1$, and less than one for $|u| < 1$. This well known fact may be shown also by the formula

$$\frac{1 - |\omega|^2}{1 - |u|^2} = \frac{1 - |a|^2}{|1 - \bar{a}u|^2} > 0. \tag{14}$$

We have therefore the following result: if for some value of a (with $|a| < 1$) the value (12) of u has a modulus greater than one, we shall have, for this same value of a,

$$|\omega| > 1, \tag{15}$$

and consequently $M(z_1, z_2) > 1$ will hold. But if, for all values of $|a| < 1$, we always have $|u| \leqq 1$, it follows that we shall always have also $|\omega| \leqq 1$, and $M(z_1, z_2) = 1$. A necessary and sufficient condition that we shall have $M(z_1, z_2) = 1$ is therefore given by the inequality

$$|z_2| \leqq |1 - \bar{z}_1 z_2 + \bar{a}z_1|, \quad \text{for} \quad |a| < 1. \tag{16}$$

This being the case, the circle around the origin with radius $|z_2|$ has no point in common with the interior of the circle of center

$1-\bar{z}_1 z_2$ and radius $|z_1|$. Conversely, the inequality (16) will hold if these circles have no common point interior to both. Consequently the condition (16) may also be written in the form

$$|z_1| + |z_2| \leqq |1 - \bar{z}_1 z_2|, \tag{17}$$

that is, in a form in which the parameter a is no longer involved.

4. *Simplification of the Condition.* This last inequality can be replaced by another which, although equivalent to it, is much simpler in form. Squaring both sides of (17), we get the condition

$$|z_1|^2 + |z_2|^2 + 2|z_1||z_2| \leqq 1 - \bar{z}_1 z_2 - z_1 \bar{z}_2 + |z_1|^2 |z_2|^2, \tag{18}$$

which is exactly equivalent to it. From the inequality

$$-\bar{z}_1 z_2 - z_1 \bar{z}_2 \leqq 2|z_1||z_2|, \tag{19}$$

which is always true for all pairs of points z_1, z_2 for which (18) (or (17)) holds, it then follows that we must have

$$(1 - |z_1|^2)(1 - |z_2|^2) \geqq 0. \tag{20}$$

It is therefore impossible that one of the points be outside and the other inside the unit circle, and a condition that none of these points be outside the unit circle is given by the inequality

$$|z_1||z_2| \leqq 1. \tag{21}$$

Adding now to the members of (18) those of the identity

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2, \tag{22}$$

and then reducing, we obtain

$$|z_1 + z_2|^2 \leqq (1 - |z_1 z_2|)^2. \tag{23}$$

If (21) holds, this is equivalent to

$$|z_1 + z_2| + |z_1 z_2| \leqq 1. \tag{24}$$

This last relation expresses therefore the condition that (17) and (21) hold simultaneously. Whenever it is satisfied, we need not state that the points z_1 and z_2 do not lie outside the unit circle. This most elegant form of the inequality (17) was pointed out to me by Szegö; it shows at first sight the symmetry in z_1 and z_2 of the condition obtained.

5. *Proof of Sufficiency of a Related Condition.* Take now two points z_1 and z_2 inside of the unit circle, for which (24) or the equivalent condition (17) holds; suppose also that

$$|a| \leqq \rho < 1. \tag{25}$$

We are going to show that under these assumptions the expression (1) cannot exceed a number which is actually less than one. We conclude first by the reasoning of §2 that any number which is not smaller than the upper bound of $|\omega(a)|$ under the condition (25) is suitable for our purpose.

Using (25) and (17), we have

$$\begin{aligned} |1 - \bar{z}_1 z_2 + \bar{a} z_1| &\geqq |1 - \bar{z}_1 z_2| - \rho |z_1| \\ &\geqq (1 - \rho)|z_1| + |z_2|, \end{aligned} \tag{26}$$

and consequently, by (12),

$$|u| \leqq \frac{|z_2|}{(1-\rho)|z_1| + |z_2|} = 1 - \frac{(1-\rho)|z_1|}{(1-\rho)|z_1| + |z_2|} < 1. \tag{27}$$

Both numbers $|a|$ and $|u|$ being not greater than one, we have the well known inequality

$$\begin{aligned} |\omega(a)| &= \left|\frac{a-u}{1-\bar{a}u}\right| \leqq \frac{|a| + |u|}{1 + |a||u|} \\ &= 1 - \frac{(1-|a|)(1-|u|)}{1+|a||u|}. \end{aligned} \tag{28}$$

Replacing in this inequality $|a|$ and $|u|$ by their upper bounds from (25) and (27), we get finally

$$|\omega(a)| \leqq 1 - \frac{(1-\rho)^2 |z_1|}{(1-\rho)|z_1| + (1+\rho)|z_2|}, \tag{29}$$

and we can therefore write

$$\left|\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right| \leqq 1 - \frac{(1-\rho)^2 |z_1|}{(1-\rho)|z_1| + (1+\rho)|z_2|}. \tag{30}$$

6. *Statement of the Theorem.* We infer from this last result that the only functions $f(z)$ of our family for which, under the assumption of (24), the expression (1) attains its maximum value *one* are precisely those which we discarded at the beginning of

§2. This completes our proof and we can state the following theorem.

THEOREM. *For every pair of points z_1, z_2 lying inside the unit circle and satisfying the condition*

$$|z_1 + z_2| + |z_1 z_2| \leqq 1, \tag{31}$$

and for every analytic function $f(z)$ which is regular for $|z| < 1$, which vanishes at the origin, and which fulfills the conditions $|f(z)| < 1$ everywhere in the circle, we always have

$$\left| \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right| < 1, \tag{32}$$

except for the case where $f(z)$ is a linear function of the form $e^{i\theta}z$.

For every pair of points z_1, z_2 inside of the unit circle for which

$$|z_1 + z_2| + |z_1 z_2| > 1, \tag{33}$$

there exist, on the contrary, analytic functions satisfying all of the above conditions for which the left-hand side of (32) has values greater than unity.

Taking $z_1 \neq 0$, and using the relation (30), we can replace the inequality (32) by another one that is more accurate. We remark for this purpose that we can take, in (30),

$$\rho = |a| = \frac{|f(z_1)|}{|z_1|}, \tag{34}$$

and that therefore we may write

$$1 - \left| \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right| \geqq \frac{(|z_1| - |f(z_1)|)^2}{|z_1|(|z_1| - |f(z_1)|) + |z_2|(|z_1| + |f(z_1)|)}. \tag{35}$$

This last inequality involves the following one, in which $|z_2|$ does not appear on the right side:

$$1 - \left| \frac{f(z_2) - f(z_1)}{z_2 - z_1} \right| > \frac{(|z_1| - |f(z_1)|)^2}{|z_1|(1 + |z_1|) + |f(z_1)|(1 - |z_1|)}, \tag{36}$$

$$1 - \left|\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right| > \frac{(|z_1| - |f(z_1)|)^2}{2|z_1|}. \tag{37}$$

Finally, if we assume that $|z_2| \leqq |z_1|$, we can replace this last inequality by the following stronger one, which we obtain directly from (35):

$$1 - \left|\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right| \geqq \frac{1}{2}\left(1 - \left|\frac{f(z_1)}{z_1}\right|\right)^2. \tag{38}$$

7. *The Geometric Meaning.* In order to find the geometric meaning of the condition (31), we take $z_1 = h$, where h is real, positive, and less than one; and we put $z_2 = x + iy$. We can then write

$$((x + h)^2 + y^2)^{1/2} \leqq 1 - h(x^2 + y^2)^{1/2}. \tag{39}$$

Squaring both sides of this last inequality, we get, after some reductions,

$$2h(x^2 + y^2)^{1/2} \leqq (1 - h^2)(1 - x^2 - y^2) - 2hx, \tag{40}$$

and this leads to the relation

$$[(1 - h^2)(1 - x^2 - y^2) - 2hx]^2 - 4h^2(x^2 + y^2) \geqq 0. \tag{41}$$

Since the expression on the left side of (41) is positive at the origin as well as on circles $x^2 + y^2 = R^2$ for large values of R, and is negative on the circle

$$(1 - h^2)(x^2 + y^2 - 1) + 2hx = 0, \tag{42}$$

the curve which is represented by

$$[(1 - h^2)(x^2 + y^2 - 1) + 2hx]^2 - 4h^2(x^2 + y^2) = 0 \tag{43}$$

consists of two loops, one of which lies inside of the circle (42) and the other outside of the same circle. On the unit circle $x^2 + y^2 = 1$, the left side of (43) has the value

$$4h^2(x^2 - 1) \leqq 0,$$

which shows that the unit circle also divides both loops.

It follows finally that the inner loop of the curve (43) divides the unit circle into two regions. In the one of these regions, which is convex, the condition $|z + h| + |zh| < 1$ is fulfilled. The

boundary of this region, which we shall call A, touches the unit circle at the point $x=-1$, $y=0$. In the second region B, which is horn-shaped, we have $|z+h|+|zh|>1$.

8. *The Curve whose Stereographic Projection is the Curve* (43). Since the curve (43) is bicircular, that is, since it has the circular points at infinity as double points, it is convenient to consider it as the stereographic projection of a spherical curve. The stereographic projection of the sphere

$$\xi^2+\eta^2+\zeta^2=\frac{1}{4} \tag{44}$$

on the plane with the coordinates x and y is defined by the formulas

$$x=\frac{\xi}{\frac{1}{2}-\zeta}, \qquad y=\frac{\eta}{\frac{1}{2}-\zeta}, \qquad x^2+y^2=\frac{\frac{1}{2}+\zeta}{\frac{1}{2}-\zeta}. \tag{45}$$

With these new variables, the equation (43) takes the form

$$h^2\xi^2+2h(1-h^2)\xi\zeta+(1-h^2+h^4)\zeta^2=\frac{h^2}{4}. \tag{46}$$

The spherical curve with which we have to deal consists therefore of the intersection of the sphere (44) with the elliptic cylinder (46).

We introduce the new rectangular coordinates ξ' and ζ' by the formulas

$$\xi=\xi'\cos\phi+\zeta'\sin\phi, \qquad \zeta=-\xi'\sin\phi+\zeta'\cos\phi, \tag{47}$$

$$\cos\phi=\frac{1}{(1+h^2)^{1/2}}, \qquad \sin\phi=\frac{h}{(1+h^2)^{1/2}}. \tag{48}$$

The equation (46) then takes the form

$$h^2\xi'^2+\frac{1}{h^2}\zeta'^2=\frac{1}{4}. \tag{49}$$

The part of this curve which corresponds to the inner loop of the bicircular curve (43) is determined by the inequalities

$$-\frac{1}{2(1+h^2)^{1/2}}\leqq\xi'\leqq\frac{1}{2(1+h^2)^{1/2}}, \qquad \zeta'\leqq\frac{-h}{2(1+h^2)^{1/2}}. \tag{50}$$

9. *Osculating Circles at the Vertices of the Curve* (43). Using these formulas, we can obtain readily all of the properties of the curve (43) which are important for the problem we are considering. Thus we can show that, when h increases from zero to one, the region A shrinks continuously; for small values of h, this region fills nearly the whole of the unit circle; if h tends towards one, this region reduces to a narrow band which surrounds the radius extending from -1 to the center.

As an example of such computations we shall determine the osculating circles at the four vertices of the boundary of the region A, that is, at the four points in which the osculating circles do not cross the curve.

Two of these points correspond to the end-points of the arc (50) of the ellipse (49). If we determine the tangents to the ellipse at these points and the segments of these tangents lying inside the circle

$$\xi'^2 + \zeta'^2 = \frac{1}{4}, \qquad \eta = 0, \tag{51}$$

the stereographic projections of these segments coincide with diameters of the osculating circles for which we are looking. We find in this way that the diameters of these circles are given by the formulas

$$-1 \leqq x \leqq \frac{1 - h - h^2}{1 + h - h^2}, \qquad y = 0, \tag{52}$$

$$-\frac{1 - h^3}{1 + h^3} \leqq x \leqq \frac{1 - h}{1 + h}, \qquad y = 0. \tag{53}$$

We can also write the equations of the circles themselves; we find

$$(1 + h - h^2)(x^2 + y^2) + 2hx - (1 - h - h^2) = 0 \tag{54}$$

for the circle with the diameter (52), and

$$(1 + h^3)(x^2 + y^2) + 2h(1 - h)(1 + h^3)x - (1 - h)^2(1 + h^2 + h^4) = 0 \tag{55}$$

for the other circle.

The two other vertices of the bicircular curve correspond to

the vertex $\xi' = 0, \zeta' = -h/2$ of the ellipse (49), and are symmetric with respect to the x axis. The osculating circle passing through that point is the stereographic projection of a circle on the sphere. Putting

$$u = \zeta' + \frac{h}{2} \tag{56}$$

and determining two suitable constants p and q, we see that this circle lies on the plane

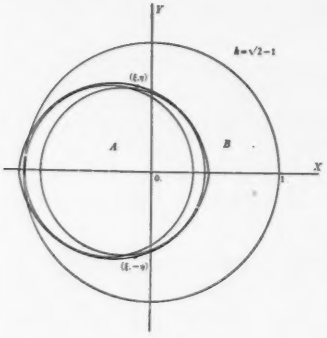

FIG. 1

$$2\eta = pu + q. \tag{57}$$

Using (56), we can write the equation of the ellipse (49) in the form

$$4h^4\xi'^2 - 4hu + 4u^2 = 0, \tag{58}$$

and we see that the intersection of the sphere (44) with the plane (57) has the form

$$4\xi'^2 + (pu + q)^2 + (2u - h)^2 = 1. \tag{59}$$

The curves (59) and (58) osculate at the point $u = 0$ if we have

$$q = \pm (1 - h^2)^{1/2}, \qquad -h^4pq = 2h(1 - h^4). \tag{60}$$

This gives for the equation of the two circles of osculation in the original variables

$$(61)\qquad (1-h^2)^{1/2}(h(x^2+y^2+1)) + (1+h^2)^{1/2}(2hx+(x^2+y^2-1)) \pm 2h^2y = 0.$$

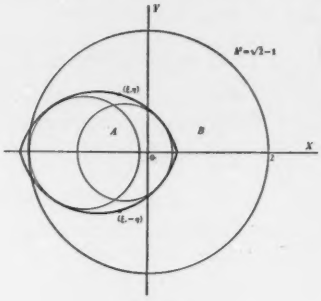

FIG. 2

It is very easy to sketch the bicircular curve which forms the boundary of the region A after the circles (54), (55), and (61) have been drawn; indeed, both circles (54) and (55) lie completely inside the region A. Both circles (61) surround A and a rather large arc on each of these circles may be considered as coinciding with the boundary of A. In Figs. 1 and 2, these circles have been drawn first for the case $h=2^{1/2}-1$, in which the point $z=h$ lies on the boundary of the region A, and secondly for $h^2=2^{1/2}-1$, in which case the circles (61) have radii of minimum value.

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A NOTE ON MATRICES DEFINING TOTAL REAL FIELDS*

BY A. A. ALBERT

Let K be algebraic of degree n over a sub-field F of the field of all real numbers. Then there is an equation

$$f(x) = x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad (a_i \text{ in } F), \tag{1}$$

which is irreducible in F, and $K = F(X)$ consists of all polynomials with coefficients in F in an algebraic quantity X for which $f(x) = 0$. We call K a *total real* field over F if the ordinary complex roots

$$x_1, \cdots, x_n \tag{2}$$

of $f(x) = 0$ are all real. The modern theory of algebraic numbers has made the study of such fields of great interest.

A particular algebraic root of $f(x) = 0$ is given by the matrix

$$Y = \begin{pmatrix} 0 & 0 \cdots 0 & -a_n \\ 1 & 0 \cdots 0 & -a_{n-1} \\ 0 & 1 \cdots 0 & -a_{n-2} \\ \cdot & \cdots\cdots & \cdot \\ 0 & 0 \cdots 1 & -a_1 \end{pmatrix}. \tag{3}$$

This is a matrix whose characteristic equation is the above $f(x) = 0$. The irreducibility of $f(x)$ implies that every n-rowed square matrix Z with elements in F and $f(x) = 0$ as characteristic equation is similar to Y, and thus every such Z defines a field $F(Z)$ equivalent to K over F.

We shall obtain a normal form here for Z such that every Z in our form and with irreducible characteristic equation defines a total real field, while conversely every total real field is defined by one of our matrices. Our result will then provide *a construction of all total real fields over F*. The irreducibility condition is of course a part of the final conditions in all problems on the construction of algebraic fields and should not be considered as

* Presented to the Society, December 31, 1936.

affecting the completeness of our criterion. We shall in fact prove the following theorem.

THEOREM. *Let D be an n-rowed diagonal matrix with positive diagonal elements in F, and S be any symmetric n-rowed square matrix with elements in F for which the characteristic function of*

$$Z = DS \tag{4}$$

is irreducible in F. Then $F(Z)$ is a total real field of degree n over F. Conversely every total real field K of degree n over F is equivalent to a field $F(Z)$ with Z given by (4).

For if D and E are the n-rowed diagonal matrices*

$$D = \operatorname{diag}\{d_1, \cdots, d_n\},\ e_i = d_i^{1/2},\ E = \operatorname{diag}\{e_1, \cdots, e_n\}, \tag{5}$$

then $D = E^2$, $E = E'$ is a real symmetric matrix,

$$E^{-1}ZE = E^{-1}E^2SE = ESE'$$

is a real symmetric matrix. Thus the characteristic roots of $E^{-1}ZE$ are all real. But they are the roots of the characteristic equation of Z and we are assuming that this equation is irreducible in F. Hence $F(Z)$ is a total real field.

Conversely let K be total real of degree n over F so that K is equivalent over F to $F(Y)$ with Y given by (3). We let V be the Vandermonde matrix

$$\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \cdot & \cdot & \cdots & \cdot \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix}. \tag{6}$$

The square of the determinant of V is the discriminant of $f(x)$ and is not zero when $f(x)$ is irreducible in F. This is our hypothesis, so that the matrix

$$T = V'V = (s_{i+j-2}), \qquad (i, j = 1, \cdots, n), \tag{7}$$

is non-singular. Also the symmetric function $s_k = \sum_{\sigma=1}^{n} x_\sigma^k$ is well known to be a polynomial in $a_1, \cdots, a_n$ with integral coeffi-

* We use the notation diag $\{d_1, \cdots, d_n\}$ to mean an n-rowed square matrix whose elements off the principal diagonal are zero and whose principal diagonal is $d_1, \cdots, d_n$.

cients, so that T has elements in F. Since V is a *real* non-singular matrix, the matrix $T = V'V$ is *positive definite* symmetric. This is actually the true reason for our result.*

There exists a non-singular B with elements in F such that

$$B'TB = \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_n \end{pmatrix}, \qquad (g_i \text{ in } F). \tag{8}$$

Since T is positive definite, so is $B'TB$, and the g_i must be positive. Thus

$$D = \operatorname{diag}\{d_1, \cdots, d_n\}, \qquad d_i = g_i^{-1} > 0, \; D^{-1} = B'TB. \tag{9}$$

By an elementary computation

$$VYV^{-1} = \operatorname{diag}\{x_1, \cdots, x_n\}. \tag{10}$$

The diagonal matrix VYV^{-1} is symmetric and

$$(VYV^{-1})' = (V')^{-1}Y'V' = VYV^{-1}, \quad (V'V)Y = Y'(V'V). \tag{11}$$

Hence $TY = Y'T$, $(B'TB)B^{-1}YB = B'Y'(B')^{-1}B'T'B$, whence

$$D^{-1}Z = Z'(D^{-1})', \qquad Z = B^{-1}YB. \tag{12}$$

The matrix $S = D^{-1}Z$ is now symmetric since $S' = Z'(D^{-1})' = S$. Then

$$Z = DS$$

as desired, and our theorem is proved.

Notice in closing that the positive elements of the matrix D are the inverses of the elements in the diagonal normal form of the *discriminant matrix* T. When this normal form of T is the identity matrix the result Z is a symmetric matrix S for which the total reality of $F(Z)$ is a classical result.†

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* See Bieberbach-Bauer, *Vorlesungen über Algebra*, 1933, p. 184, for the known theorem stating that T is positive definite when $f(x) = 0$ has all real roots. That this result is true is an evident consequence of the definition of positive definiteness.

† See L. E. Dickson, *Modern Algebraic Theories*, 1926, p. 76; Theorem 12.

THE CATEGORY AND BOREL CLASS OF CERTAIN SUBSETS OF $\mathcal{L}_p$*

BY J. C. OXTOBY

We consider the subsets of the Lebesgue spaces $\mathcal{L}_p([0, 1])$, $(p \geqq 1)$, corresponding to some familiar classes of functions, and show that they are Borel sets and of first category. The method consists in utilizing the boundedness properties of the functions to obtain representations of the sets in question in terms of closed sets, and in each case gives an evaluation of the Borel class.† By the set corresponding to a class is meant the set consisting of these and all equivalent functions.

LEMMA. *The class of functions f in $\mathcal{L}_p$ such that $f(x) \leqq g(x)$ for almost all x in $[a, b]$, $(0 \leqq a < b \leqq 1)$, g being an arbitrary function in $\mathcal{L}_p$, constitutes a nowhere dense closed set in $\mathcal{L}_p$.*

Consider any sequence in the set, converging in $\mathcal{L}_p$ to f. It is possible to pick a subsequence that converges pointwise to $f(x)$ almost everywhere.‡ It follows that $f(x) \leqq g(x)$ almost everywhere in $[a, b]$, which shows that the set is closed. To prove that it is nowhere dense, it now suffices to show it is included in the derived set of its complement. Let f be any function in the set. Changing its value to $g(x)+1$, say, on a small subinterval of $[a, b]$, we can obtain a function $f_\epsilon(x)$, in the complementary set, such that $\|f_\epsilon - f\| < \epsilon$. The lemma evidently remains true for the class of functions bounded *below* by $g(x)$ almost everywhere in $[a, b]$.

THEOREM 1. *The set C of points of $\mathcal{L}_p$ corresponding to continuous functions is an $F_{\sigma\delta}$ set of first category.*§

* Presented to the Society, September 5, 1936.

† For a general method of evaluating the Borel class of a set see Kuratowski, Fundamenta Mathematicae, vol. 17 (1931), pp. 249–272. For a specific result concerning the category of a set in $\mathcal{L}_p$, see E. S. Quade, this Bulletin, vol. 41 (1935), pp. 83–84.

‡ Hobson, *The Theory of Functions of a Real Variable*, vol. 2, second edition, 1926, p. 245.

§ An F_σ set is a denumerable union of closed sets. An $F_{\sigma\delta}$ set is a denumerable intersection of F_σ sets. A denumerable union of nowhere dense sets is said to be of first category.

Let (g_m, h_m) be an enumeration of all pairs of rational step-functions, g_m lower, h_m upper semi-continuous. Let E_{nm} be the class of all functions f in $\mathcal{L}_p$ such that $g_m(x) \leqq f(x) \leqq h_m(x)$ almost everywhere in $[0, 1]$, and $0 \leqq h_m(x) - g_m(x) \leqq 1/n$ in $[0, 1]$. E_{nm} is closed and nowhere dense, since it is the intersection of two sets of the type considered in the lemma. We have the representation $C = \prod_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{nm}$. For, by the Heine-Borel theorem, any continuous function can be enclosed between two rational step-functions, conveniently taken semi-continuous, differing uniformly by less than $1/n$; and conversely, a function so enclosed almost everywhere is equivalent to a continuous function, the common limit of the bounding step-functions.

THEOREM 2. *The sets S_u and S_l of points of $\mathcal{L}_p$ corresponding to upper and lower semi-continuous functions are $F_{\sigma\delta}$ sets of first category.*

Let $\{h_m\}$ be an enumeration of all upper semi-continuous rational step-functions. Denote by E_{nm} the set of functions f in $\mathcal{L}_p$ such that $f(x) \leqq h_m(x)$ almost everywhere in $[0, 1]$, and $\int_0^1 |f(x) - h_m(x)|\, dx \leqq 1/n$. This is the intersection of a set of the type considered in the lemma with the set characterized by the second condition. To show that the latter is closed, consider any sequence $\{f_k\}$ in it converging in $\mathcal{L}_p$ to f. Then f_k and f are in $\mathcal{L}_1$, and, by the triangle and Hölder inequalities, we have

$$\int_0^1 |f - h_m|\, dx \leqq \int_0^1 |f_k - h_m|\, dx + \int_0^1 |f_k - f|\, dx$$

$$\leqq \frac{1}{n} + \left(\int_0^1 |f_k - f|^p dx \right)^{1/p}.$$

Letting $k \to \infty$, we see that the set is closed. It follows that E_{nm} is closed and nowhere dense. We have $S_u = \prod_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{nm}$. For, by considering upper Darboux sums, an upper semi-continuous function is seen to be bounded above by step-functions, which may be taken rational, with integrals arbitrarily close to that of the function.* On the other hand, a function in $\mathcal{L}_p$ that can

* Carathéodory, *Vorlesungen über reelle Funktionen*, second edition, 1927, p. 457. To apply this theorem it is sufficient that the function be bounded above. In our case, this follows from the upper semi-continuity, since we are considering the *closed* interval $[0, 1]$.

be so approximated almost everywhere is equivalent to the lower bound of the approximating sequence, an upper semi-continuous function. The theorem for S_l follows from the homeomorphism $Tf=-f$.

THEOREM 3. *The set R of points of $\mathcal{L}_p$ corresponding to bounded Riemann integrable functions is the intersection of S_u and S_l and hence is an $F_{\sigma\delta}$ set of first category.*

Every bounded Riemann integrable function is in $\mathcal{L}_p$ and continuous almost everywhere, hence equivalent to its maximum and minimum functions, respectively upper and lower semi-continuous.* On the other hand, two equivalent functions, respectively upper and lower semi-continuous, are continuous wherever equal, that is, almost everywhere. They are also bounded, hence Riemann integrable.

It is interesting to note that whereas the intersection of the *classes* of upper and lower semi-continuous functions is the class of continuous functions, nevertheless the intersection of the corresponding *point sets* in $\mathcal{L}_p$ is not the set corresponding to continuous functions, but the larger set R.

THEOREM 4. *If $q>p$, then $\mathcal{L}_q$ is contained in $\mathcal{L}_p$ and is an F_σ set of first category in $\mathcal{L}_p$.*†

Denote by the symbol E_N the set of functions in $\mathcal{L}_p$ such that $\int^1|f(x)|^q dx \leqq N$. Consider any sequence in E_N with limit in $\mathcal{L}_p$. We may suppose the sequence pointwise convergent almost everywhere. By truncating the functions to the bound m, applying the Lebesgue convergence theorem, and letting $m\to\infty$, the set E_N is seen to be closed. To show that it is also nowhere dense, consider any f in E_N, $\epsilon>0$. Choose $\eta>2\epsilon$ such that $\epsilon^p|\eta-2N|^q>N\eta^p$. Then choose a subset $\mathcal{E}$ of $[0, 1]$ of measure $(\epsilon/\eta)^p$ on which $|f(x)|\leqq 2N$. This is possible, since for a function in E_N, $|f(x)|\leqq 2N$ on a set of measure at least $1/2$, and

* Hobson, op. cit., vol. 1, third edition, 1927, pp. 465, 308.

† This theorem, at least as regards category, was proved earlier by Garrett Birkhoff and E. S. Quade, but was not published. The evaluation of the Borel class as F_σ is here the precise determination, since the set is not closed, and neither is it G_δ. See Mazur and Sternbach, Studia Mathematica, vol. 4 (1934), p. 50, where it is shown that a linear G_δ set is necessarily closed.

η was so chosen that $(\epsilon/\eta)^p<1/2$. Define $f_\epsilon(x)=f(x)+\eta$ on $\mathcal{E}$ and equal to $f(x)$ elsewhere. Then $\|f_\epsilon-f\|=\epsilon$, while

$$\int_0^1 |f_\epsilon(x)|^q dx > N.$$

Since $\mathcal{L}_q=\sum_{N=1}^{\infty}E_N$, the theorem is established.*

THEOREM 5. *The class of functions in $\mathcal{L}_p$ bounded away from some value almost everywhere in a neighborhood of some point of $[0, 1]$ constitutes an F_σ set of first category.*

Here we let $E(r_1, r_2, r_3, r_4)$ be the nowhere dense closed set consisting of all functions f in $\mathcal{L}_p$ such that $f(x)\leqq r_3$ or $f(x)\geqq r_4$, $(r_3<r_4)$, for almost all x in $r_1\leqq x\leqq r_2$, $(0\leqq r_1<r_2\leqq 1)$. The set in question is obtained by summing over all rational r_1, r_2, r_3, r_4 satisfying the stated inequalities.

The methods used in proving the above theorems can be adapted to treat questions of relative category. For example, we have the following theorem.

THEOREM 6. *Relative to the complete subspace $B_{M,p}$ consisting of all functions f in $\mathcal{L}_p$ such that $|f(x)|\leqq M$, $(0\leqq x\leqq 1;\ M>0)$, the sets $CB_{M,p}$, $S_uB_{M,p}$, $S_lB_{M,p}$, $RB_{M,p}$, are $F_{\sigma\delta}$ of first category. Except for a set of first category, every function in $B_{M,p}$ has upper and lower Riemann integrals equal to M and $-M$, respectively.*

The proof consists in showing that the set of functions in $B_{M,p}$ bounded away from M or $-M$ almost everywhere in some subinterval forms a set of first category in $B_{M,p}$, and in noting that a function in any one of the sets mentioned in the theorem either has this property or is equivalent to one of the constant functions M, $-M$.

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* The latter half of this proof can be avoided by appealing to a theorem of Banach, *Théorie des Opérations Linéaires*, 1932, p. 36, according to which a proper linear subset of $\mathcal{L}_p$ is necessarily of first category if it is a Borel set. This remark applies also to the sets C and R.

A GENERALIZED ELEMENT OF DECOMPOSITION FOR DOUBLY PERIODIC FUNCTIONS*

BY G. D. NICHOLS

1. *Introduction.* Hermite† has defined a function $f(x)$ to be doubly periodic of the third kind if it is meromorphic and satisfies two periodicity relations of the form

$$f(x + 2K) = e^{\alpha x+\beta} f(x),$$
$$f(x + 2iK') = e^{\gamma x+\delta} f(x).$$

These relations may be transformed into the following ones

$$f(x + \pi) = f(x),$$
$$f(x + \pi\tau) = e^{2i(ax+b)} f(x), \tag{1}$$

and it has been shown that the constant a in (1) is an integer and is equal to the excess of the number of poles over the number of zeros of $f(x)$ in a primitive period cell. For our purposes in the following it is helpful to consider elliptic functions and doubly periodic functions of the second kind as special cases of the above, namely, when $a=b=0$, and $a=0$, respectively.

We derive a generalized element of decomposition which by proper specialization of the constants can be used equally well to obtain expansions for doubly periodic functions of the first, second, or third kinds. It should be noted, however, that the method of the present paper fails to yield a decomposition in the case of doubly periodic functions of the third kind having more zeros than poles, that is, when $a<0$, since the convergence of the series representing the integrals for C_r in the following is then destroyed. This same difficulty has been met by other writers in discussing the decomposition of doubly periodic functions of the third kind.‡ However, Appell has solved this difficulty by interchanging the roles of x and y in his element of decomposition obtained for the case where a is positive.

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* Presented to the Society, April 11, 1936.

† Hermite, Oeuvres, vol. 3, p. 329 (footnote); Appell, Mémorial des Sciences Mathématiques, fascicule 36 (1929).

‡ See, for example, a paper by Basoco in Acta Mathematica, vol. 57.

The derivation starts from the integral first used by Teixeira* in obtaining a formula for the expansion of doubly periodic functions of the second kind. The method differs from those frequently used in that it is direct and does not assume a priori the existence and form of the desired element of decomposition.

2. *Element of Decomposition.* Let $f(x)$ be a doubly periodic function satisfying relations (1), where the imaginary part of τ is positive. To simplify the notation we shall first assume that $f(x)$ has a single pole of order k in each period parallelogram. Let $x=y$ be the affix of this pole in the period parallelogram $ABCD$, where AB coincides with the axis of reals and CD is above AB. Then in the neighborhood of this pole,

$$f(x) = \frac{A_k}{(x-y)^k} + \cdots + \frac{A_2}{(x-y)^2} + \frac{A_1}{(x-y)} + P(x-y).$$

Consider now Teixeira's function,

$$\Phi(t) = \frac{e^{2it}}{e^{2it} - e^{2iz}} f(t).$$

Taking the integral of $\Phi(t)$ around $ABCD$ and applying Cauchy's theorem, we obtain the formula

$$(2) \qquad f(x) = \sum_{r=-\infty}^{\infty} C_r e^{2rix} - iA_1 + \sum_{s=1}^{k} D_y^{(s-1)} \operatorname{ctn}(x-y) \frac{A_s}{(s-1)!},$$

where

$$(3) \qquad C_r = \frac{1}{\pi} \int_{AB} f(t) e^{-2rit} dt, \qquad (r = 0, 1, 2, \cdots),$$

$$C_{-r} = \frac{1}{\pi} \int_{DC} f(t) e^{2rit} dt, \qquad (r = 1, 2, 3, \cdots).$$

To calculate C_r, $r>0$, integrate $f(t)\ e^{-2rit}$ around a sequence of n period parallelograms of width π, having AB for upper

* Teixeira, Journal für Mathematik, vol. 125 (1901), pp. 301–318; G. D. Nichols, Tôhoku Mathematical Journal, vol. 37 (1934), pp. 252–258; M. A. Basoco, this Bulletin, vol. 37 (1931), pp. 111–124, and vol. 38 (1932), pp. 560–568.

boundary and extending in the downward direction. Applying Cauchy's theorem and allowing n to become infinite gives

$$(4) \qquad C_r = -2i\sum_{s=1}^{k}\sum_{n=1}^{\infty} e^{-2i(ayn+yr+nb)}q^{an(n+1)+2rn}(-2ir-2ian)^{s-1}\frac{A_s}{(s-1)!},$$

where $q=e^{\pi i\tau}$. The coefficient C_{-r} is obtained by integrating $f(t)e^{2\pi it}$ around a similar sequence of parallelograms having CD for lower boundary and extending upward. Its value is obtained from that for C_r by replacing i with $-i$. For elliptic functions C_0 cannot be calculated by this method, but for the other two kinds of functions its value is given by (4). The regions of convergence for the C's will be discussed below in connection with the final result.

It is interesting to note at this point that if we had integrated $f(t)e^{-2\pi it}$ around the period parallelogram $ABCD$, we would have been led to the difference equation

$$C_r - q^{-2r}e^{2ib}C_{r-a} = 2ie^{-2iyr}\sum_{s=1}^{k}(-2ir)^{s-1}\frac{A_s}{(s-1)!},$$

of which the above value of C_r is a solution. The value of C_{-r} could have been obtained in a similar manner.

Substituting the above values for the C's into (2) gives

$$f(x) = C_0 + \sum_{n=-\infty}^{\infty}\sum_{s=1}^{k} q^{an(n-1)}e^{2nib}D_y^{(s-1)}e^{2iyan} \cdot [\operatorname{ctn}(x-y-n\pi\tau)-\alpha i]\frac{A_s}{(s-1)!},$$

where $\alpha=1$ for n zero or positive, $\alpha=-1$ for n negative, and $-I(\pi\tau)<I(x-y)<I(\pi\tau)$.

We may now write our final result as follows:

$$(5) \qquad f(x) = B + \sum\frac{\phi_k(x,y,b)}{(k-1)!},$$

where

$$\phi_k(x,y,b) = A_1\psi(x,y,b) + A_2D_y\psi(x,y,b) + \cdots + A_kD_y^{(k-1)}\psi(x,y,b),$$

and where the summation is extended over all poles of $f(x)$ in a given period parallelogram and where

$$(6)\quad \psi(x, y, b) = \sum_{n=-\infty}^{\infty} q^{an(n-1)} e^{2nib} e^{2iyan} [\operatorname{ctn}(x - y - n\pi\tau) - \delta i]$$

is the desired generalized element of decomposition.

For elliptic functions we take $a = b = 0$, $\delta = 0$ for n zero, $\delta = 1$ for n positive, and $\delta = -1$ for n negative. B is the usual constant occurring in classical formulas of decomposition and our psi-function (6) is seen to be identical with the zeta-function of Jacobi and Hermite.*

For doubly periodic functions of the second kind $a = B = 0$, and $\delta = 1$ for all values of n. In this case there is the further restriction that $0 > I(b) > -I(\pi\tau)$. The element of decomposition obtained from (6) for this class of functions does not seem to have been given hitherto, although it can be checked by means of certain of the results given by Teixeira.† By a slight change in procedure a similar element can be derived which is valid for $I(b)$ positive. For doubly periodic functions of the third kind $b = B = 0$ and $\delta = 1$ for all values of n. Our function (6) for this case is closely related to the chi-function of Appell,‡ and can be shown to give the same expansion for $f(x)$. To do this consider $g(x) = \sum_{n=-\infty}^{\infty} q^{an(n-1)} e^{2izan}$, which satisfies the two relations $g(x+\pi) = g(x)$, $g(x+\pi\tau) = e^{-2iaz} g(x)$. Hence $f(x) \cdot g(x)$ is an elliptic function and furthermore has the same poles as $f(x)$. The residue at the pole $x = y$ is

$$\sum_{s=1}^{k} D_y^{(s-1)} g(y) \frac{A_s}{(s-1)!},$$

and the sum of all such residues at poles within each period cell is zero. But this sum is exactly the part added to the expansion (5) by the difference between the element (6) and that of Appell.

UNIVERSITY OF ARKANSAS

* Appell and Lacour, *Principes des Fonctions Elliptiques*, 2d edition, 1922, p. 115.

† Teixeira, loc. cit.

‡ Appell and Lacour, *Principes des Fonctions Elliptiques*, 2d edition, 1922, p. 396; Halphen, *Fonctions Elliptiques*, vol. 1, p. 468; M. A. Basoco, Acta Mathematica, vol. 57, pp. 95–100.

ERRATA IN THE LEHMER FACTOR STENCILS

BY J. D. ELDER

In his factor stencils, D. N. Lehmer has made available a powerful tool in factorization problems. However, they present in use certain minor disadvantages. It was suggested to me that possibly these could be eliminated by using Hollerith cards. An additional advantage would be the ease of reproduction. I constructed with these cards a small set of factor stencils equivalent to the first eight columns of the Lehmer stencils. (As first constructed these were merely a partial copy of the Lehmer stencils.) These were so perfect mechanically that a thorough comparison of different stencils was extremely easy. A comparison of the new stencils for $+R$ and $-R$, for various prime values of R showed contradictions, and further investigations convinced me of the desirability of a complete recalculation of the Lehmer stencils.

This is now well under way and has been completed for the first eight columns. The number of errors found is larger than anticipated. It seems desirable to make this information available, as it will take the better part of a year to complete the entire calculation. Professor Lehmer has graciously loaned the original stencils to me for purposes of comparison; in a few entries these differ from the copy which was first used.

In the following table the number in the L-column indicates the number of the cell considered; a $+$ or $-$ indicates, respectively, that the cell in question is or is not punched. C refers to the copy of Lehmer's stencil at the University of Michigan, O to the original stencil, and K to the correct entries. The entry for C is given only if it differs from the corresponding O entry.

A comparison of the following table and the original stencils disclosed the significant fact that almost all of the errors found had been made after the direct calculation for prime values of R was completed. I had replaced Professor Lehmer's visual method of constructing stencils for composite R's by a photographic method which eliminates some of the possible sources of error; it has the additional advantage of requiring considerably less time.

R	*L*	*K*	*C*	*O*	*R*	*L*	*K*	*C*	*O*	*R*	*L*	*K*	*C*	*O*
13	3	+		−	127	14	+		−	182	601	+		−
19	302	+	−	+	127	138	+		−	183	31	−		+
22	689	+		−	127	427	+		−	186	31	−		+
26	3	−		+	127	743	+		−	186	286	−		+
31	199	+		−	130	3	+		−	187	10	+		−
38	8	+		−	131	3	−		+	187	474	+		−
38	390	+		−	131	159	+		−	191	511	+		−
38	403	+		−	131	276	+		−	193	3	+		−
38	689	+		−	134	101	+		−	195	31	+		−
42	131	+		−	134	689	+		−	195	88	+		−
42	253	+		−	138	31	+		−	199	200	+		−
42	286	+		−	138	286	−		+	199	436	+		−
46	689	+		−	138	308	+	−	+	203	145	+		−
51	500	+		−	142	689	+		−	205	100	−		+
62	689	+		−	143	3	−		+	206	416	+		−
65	3	−		+	146	94	+		−	209	294	+		−
65	744	+		−	151	54	+		−	210	252	+		−
66	286	+		−	151	260	+		−	211	480	+		−
70	689	+		−	151	272	+		−	214	274	+		−
78	30	+		−	151	354	+		−	214	278	+		−
86	689	+		−	151	499	+		−	214	689	+		−
91	3	+		−	157	40	+		−	221	3	−		+
94	689	+		−	157	720	−		+	222	30	+		−
102	30	+		−	157	740	−		+	223	184	+		−
102	500	+		−	158	19	+		−	223	476	+		−
110	511	+		−	158	70	+		−	223	528	+		−
111	4	+		−	159	31	−		+	226	101	+		−
111	31	+		−	166	3	+		−	227	476	+		−
114	21	+		−	166	33	+		−	227	528	+		−
114	31	+		−	166	45	+		−	231	31	−		+
114	286	−		+	170	10	+		−	235	100	+		−
118	466	+		−	170	11	+		−	237	607	+		−
118	689	+		−	179	475	+	−	+	238	71	+		−
123	31	−		+	182	3	−		+					
−2	286	+	−	+	−102	31	+		−	−194	512	+		−
−2	428	+	−	+	−102	286	+		−	−195	689	+		−
−10	689	+		−	−106	689	+		−	−199	200	+		−
−14	404	+		−	−110	474	+		−	−201	31	+		−
−14	689	+		−	−110	689	+		−	−206	73	+		−
−17	500	+		−	−114	30	+		−	−206	74	+		−
−26	3	+		−	−119	500	+		−	−206	180	+		−
−26	404	+		−	−122	689	+		−	−206	384	+		−
−26	689	+		−	−129	31	+		−	−209	474	+		−
−26	708	+		−	−130	3	−		+	−210	11	+		−
−29	72	+		−	−131	3	+		−	−210	29	+		−
−30	708	+		−	−138	30	+		−	−210	252	+		−

R	*L*	*K*	*C*	*O*	*R*	*L*	*K*	*C*	*O*	*R*	*L*	*K*	*C*	*O*
−34	500	+		−	−141	31	+		−	−213	31	+		−
−35	101	+		−	−141	688	+	−	+	−214	100	+		−
−37	3	−		+	−142	576	+	−	+	−214	176	+		−
−38	303	+		−	−143	3	+		−	−215	100	+		−
−39	509	+		−	−157	720	−		+	−215	161	+		−
−41	101	+		−	−157	740	+		−	−218	689	+		−
−62	13	+		−	−158	19	+		−	−221	3	+		−
−65	3	+		−	−165	31	+		−	−221	476	+		−
−66	30	+		−	−166	45	+		−	−221	500	+		−
−69	42	+		−	−166	147	+		−	−221	528	+		−
−74	4	+		−	−170	500	+		−	−222	31	+		−
−74	689	+		−	−170	689	+		−	−222	100	+		−
−77	27	+		−	−173	46	+	−	+	−222	286	+		−
−78	31	−		+	−174	31	+		−	−226	101	+		−
−78	286	+		−	−174	286	+		−	−229	528	+		−
−85	500	+		−	−177	31	+		−	−231	31	−		+
−86	73	+		−	−181	45	+		−	−235	161	+		−
−86	86	+		−	−182	3	+		−	−237	31	+		−
−91	3	−		+	−187	500	+		−	−238	500	+		−
−93	31	−		+	−190	11	+		−	−238	689	+		−
−93	184	+		−	−193	3	−		+					

Note by the Editors. This table has been examined by D. N. Lehmer and is thought by him to be correct. It has been re-checked by the author independently from the galley proofs, so that there should be no typographical errors. THE EDITORS.

UNIVERSITY OF MICHIGAN

AN APPLICATION OF DERIVATIVES OF NON-ANALYTIC FUNCTIONS IN PLANE STRESS PROBLEMS*

BY V. P. JENSEN AND D. L. HOLL

1. *The Plane Stress Problem.* A plane state of stress is defined in a region in the xy plane by an Airy's stress function $F(x, y)$, where F satisfies

$$\nabla^4 F(x, y) = \nabla^2 \nabla^2 F = \nabla^2 (F_{xx} + F_{yy}) = 0. \tag{1}$$

The normal stresses σ_x and σ_y in the directions of x and y, respectively, and the corresponding shearing stress τ_{xy} are obtained from F when no body forces are present by the following:†

$$\sigma_x = F_{yy}, \qquad \sigma_y = F_{xx}, \qquad \tau_{xy} = -F_{xy}. \tag{2}$$

Equilibrium conditions show that the stress tensor at any point may be referred to any set of orthogonal planes by the relations

$$\begin{aligned} \sigma_{x'}, \sigma_{y'} &= \frac{\sigma_x + \sigma_y}{2} \pm \frac{\sigma_x - \sigma_y}{2} \cos 2\theta \pm \tau_{xy} \sin 2\theta, \\ \tau_{x'y'} &= \frac{\sigma_y - \sigma_x}{2} \sin 2\theta + \tau_{xy} \cos 2\theta, \end{aligned} \tag{3}$$

where the $x'y'$ axes have been rotated through the positive angle θ from the xy axes.

2. *The Stress Circle and Kasner's Derivative Circle.* A graphical construction due to Mohr‡ is frequently employed in place of equations (3). In the complex plane $\gamma = \sigma + i\tau$, describe a circle having its center on the σ axis and passing through the points (σ_x, τ_{xy}) and $(\sigma_y, -\tau_{xy})$ which are designated respectively as points C and E. Then, corresponding to a counter-clockwise ro-

* Presented to the Society, November 28, 1936.

† Subscripts on F denote partial derivatives with respect to the indicated variables. The subscripts on the stresses σ and τ refer to directions along which the stresses act.

‡ O. Mohr, *Abhandlungen aus dem Gebiete der Technische Mechanik*, 2d edition, 1914.

tation θ of the x and y axes, the diameter CE rotates clockwise through an angle 2θ into a new position $C'E'$. The points C' and E' then have the new coordinates $(\sigma_{x'}, \tau_{x'y'})$ and $(\sigma_{y'}, -\tau_{x'y'})$ defined in (3).

For every point in a stressed region there exists a Mohr's dyadic stress circle which describes the complete state of stress at that point. One observes that as θ varies in the xy plane, a characteristic point E in the γ plane moves around the circle at twice the rate at which θ changes and in the opposite sense.

It may be recalled that the property ascribed to E is identical with that given by Kasner* for γ, where γ is defined as the directional derivative of a non-analytic function of a complex variable. The circle described by γ has been called by Hedrick† the Kasner circle. Through the similarity of properties of the Kasner circle for the directional derivative and of Mohr's circle for a state of stress, one is led to seek the function of a complex variable which will lead through its directional derivatives to a family of Mohr's circles.

3. *Properties of the $H(z, \bar{z})$ Function.* Let the Airy's function $F(x, y)$ be written

$$F(x, y) = F\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = F(z, \bar{z}). \tag{4}$$

Equations (1) and (2) become

$$F_{zzzz} = 0, \tag{5}$$

$$\sigma_x, \sigma_y = 2F_{z\bar{z}} \mp (F_{zz} + F_{\bar{z}\bar{z}}), \qquad \tau_{xy} = -i(F_{zz} - F_{\bar{z}\bar{z}}). \tag{6}$$

Let the function H and its directional derivative be defined by

$$H(z, \bar{z}) = 2F_{\bar{z}}, \tag{7}$$

$$\gamma_H = H_z + H_{\bar{z}}e^{-2i\theta} = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2} + i\tau_{xy}\right)e^{-2i\theta}. \tag{8}$$

It is evident from (8) that γ_H represents the point E' on Mohr's circle and that the congruence of Kasner circles for the

* Kasner, Science, vol. 66 (1927), pp. 581–582.

† E. R. Hedrick, *Non-analytic functions of a complex variable*, this Bulletin, vol. 39 (1933).

function H is identical with the family of Mohr's circles for the stresses defined by F. Furthermore, the *principal directions* and *characteristic curves*, as defined for the non-analytic function H, coincide respectively with the directions of the principal stresses and the stress trajectories in the stress field of F.

The significance of γ_H, determined graphically from (8), may also be determined algebraically by solving (3) for σ_x, σ_y, and τ_{xy} in terms of $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$. Substituting these results in (8), one obtains the relations

$$(9) \qquad \gamma_H = \sigma_{y'} - i\tau_{x'y'}, \qquad i\gamma_H = \gamma_{iH} = \tau_{x'y'} + i\sigma_{y'}.$$

The following properties are immediately deducible:

A. *The directional derivative of iH, taken tangent to any arc, is the resultant unit stress acting upon that arc.*

B. *The directional derivative of H, taken tangent to a stress trajectory, is real; conversely, if it is real the arc is acted upon by normal stresses only.*

C. *The directional derivative of H, taken tangent to a stress-free boundary, vanishes everywhere along that boundary.*

These properties are useful in discussing the nature of stresses on curvilinear boundaries. Furthermore, there is a physical reason for taking the directional derivatives of second and higher orders in such a manner that the slope of the curve of approach remains constant at any point while the order of the derivative increases. As pointed out by Hedrick,* this is only one of the possible choices of a definition for derivatives of second and higher orders.

4. *Application to Conformal Mapping.* Let a given region in the complex plane $w=u+iv$, having stresses defined by an Airy's function $\mathfrak{F}(w, \bar{w})$, be mapped conformally into a region in the z plane by

$$(10) \qquad z = g(w), \quad \text{with } h = \left|\frac{dz}{dw}\right| = (g'\bar{g}')^{1/2}, \quad \text{and } g' = \frac{dg}{dw},$$

and where $\bar{g}'$ is formed from g' by replacing i by $-i$. At a given point $w=w_1$ on the boundary of the stressed region in the w plane, let the tangent to the boundary make an angle θ_1 with

* Loc. cit.

the line $v=$constant drawn through w_1. When the point w_1 maps into z_1, the tangent to the new boundary at z_1 makes an angle $\theta=\theta_1+\theta_2$ with the line $y=$constant. Since the mapping is conformal, θ_2 is the angle between the line $y=$constant and the tangent at z_1 to the mapped arc of $v=$constant. Then

$$g' = he^{i\theta_2}. \tag{11}$$

If the original boundary is free from stress, then condition C applies in the w plane and may be written

$$\gamma_{\mathfrak{IC}}\Big]_{\theta=\theta_1} = \mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{\bar{w}\bar{w}}e^{-2i\theta_1} = 0. \tag{12}$$

Since both the real and imaginary parts of (12) must be zero, the difference of the real and imaginary parts becomes

$$\mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{ww}e^{2i\theta_1} = 0. \tag{13}$$

Multiplication of (12) and (13) by $e^{-2i\theta_1}$ yields two similar restrictions on $\mathfrak{F}$ at the stress-free boundary.

As an illustration of the use of the foregoing, consider Michell's* problem of the inversion of a field of stress. The mapping function is

$$zw = 1, \tag{14}$$

with the additional relation that the new stress function, $F(z, \bar{z})$, applicable to the new region in the z plane is given by

$$F(z, \bar{z}) = z\bar{z}\mathfrak{F}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right) = z\bar{z}\mathfrak{F}(w, \bar{w}). \tag{15}$$

By means of the non-analytic functions

$$H = 2F_z, \qquad \text{and} \qquad \mathfrak{IC} = 2\mathfrak{F}_{\bar{w}},$$

the results of Michell may be obtained, namely, a stress-free boundary maps into a boundary acted upon by a constant normal stress, and stress trajectories map into stress trajectories.

The directional derivative of H, taken along a boundary in the z plane, is

* J. H. Michell, *The inversion of plane stress*, Proceedings of the London Mathematical Society, vol. 34 (1901).

$$\gamma_H\Big]_{\theta=\theta_1+\theta_2} = 2\left[F_{zz} + F_{\bar{z}\bar{z}}e^{-2i(\theta_1+\theta_2)}\right]$$

$$(16) \qquad = \frac{2}{h}\left[-h\mathfrak{F} + \frac{1}{\bar{w}}\mathfrak{F}_w + \frac{1}{w}\mathfrak{F}_{\bar{w}} - (\mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{\bar{w}\bar{w}}e^{-2i\theta_1})\right]$$

$$= 2(-\mathfrak{F} + w\mathfrak{F}_w + \bar{w}\mathfrak{F}_{\bar{w}}) - \frac{2}{h}\left[\gamma_{\mathfrak{IC}}\right]_{\theta=\theta_1}.$$

This equation is obtained by using (14), (15) and the operators

$$h^2F_{z\bar{z}} = F_{w\bar{w}}, \qquad (\bar{g}')^3F_{\bar{z}\bar{z}} = \bar{g}'F_{\bar{w}\bar{w}} - \bar{g}''F_{\bar{w}}.$$

From the boundary condition (12), equation (16) yields

$$(17) \qquad \gamma_H\Big]_{\theta=\theta_1+\theta_2} = 2(-\mathfrak{F} + w\mathfrak{F}_w + \bar{w}\mathfrak{F}_{\bar{w}}) = \text{a real function},$$

since $\mathfrak{F}$ is real. Therefore, from the condition B, the boundary is acted upon by normal stresses only.

To determine the nature of the normal stresses on the new boundary, one finds the directional derivative of (17) along the boundary to be

$$(18) \qquad \frac{d\gamma_H}{dz}\Big]_{\theta=\theta_1+\theta_2} = -2w^3(\mathfrak{F}_{ww} + \mathfrak{F}_{w\bar{w}}e^{-2i\theta_1})$$

$$- 2\bar{w}w^2(\mathfrak{F}_{w\bar{w}} + \mathfrak{F}_{\bar{w}\bar{w}}e^{-2i\theta_1}) = 0.$$

Equation (18) is obtained from (17) by transforming from the independent variables z and $\bar{z}$ to w and $\bar{w}$. The boundary conditions similar to (12) and (13) show that the directional derivative vanishes on the boundary, and thus the normal stresses which act upon the transformed boundary must be constant.

In the w plane let θ_1 now designate the angle between the u axis and the tangent to a stress trajectory at any point w_1 within the stressed region. Then, by condition B, the directional derivative of $\mathfrak{IC}$ along the stress trajectory, $\gamma_{\mathfrak{IC}}]_{\theta=\theta_1}$ is real; hence, by (16), at any point z_1 on the mapped trajectory, the directional derivative, $\gamma_H]_{\theta=\theta_1+\theta_2}$, is also real. Therefore, by the second part of condition B, the original stress trajectories map into stress trajectories for the new stress field.

IOWA STATE COLLEGE

ON CERTAIN CONFIGURATIONS OF POINTS IN SPACE AND LINEAR SYSTEMS OF SURFACES WITH THESE AS BASE POINTS*

BY ARNOLD EMCH

1. *Introduction.* Configurations of this sort in connection with certain surfaces are known in large numbers. For example, the vertices of the 45 triangles formed by the 27 lines on a general cubic surface; the 12 vertices of 3 desmic tetrahedra; the 24 double points of the 6 quintic cycles of the symmetric collineation group on five variables interpreted in S_3; the G_{18} group of points which I found on a new normal form of the cubic surface,† and so on.

In this paper I shall establish two new configurations of points and investigate their properties and some of the surfaces on these points.

2. *The G_{27} of W-Points.* This configuration is defined by the system of points W

$$(1) \quad W = (\omega^\alpha, \omega^\beta, \omega^\gamma, 1), \quad \omega^3 = 1, \quad \alpha, \beta, \gamma \equiv 0, 1, 2, (\text{mod } 3),$$

which yields the group G_{27} of 27 points W. Consider now any of the W's and two more of the set as follows:

$$W_0 = (\omega^\alpha, \quad \omega^\beta, \quad \omega^\gamma, \quad 1),$$
$$W_1 = (\omega^{\alpha+1}, \omega^{\beta+1}, \omega^{\gamma+1}, 1),$$
$$W_2 = (\omega^{\alpha+2}, \omega^{\beta+2}, \omega^{\gamma+2}, 1).$$

Subtracting corresponding coordinates of these three points, say $(W_0 - W_1)$, $(W_1 - W_2)$, $(W_2 - W_0)$, and dividing in each case by $(1-\omega)$, we obtain the point $V(\omega^\alpha, \omega^\beta, \omega^\gamma, 0)$. The cross-ratio of the four points is

$$(VW_0W_1W_2) = (\infty, \omega^\alpha, \omega^{\alpha+1}, \omega^{\alpha+2}) = (\infty, 1, \omega, \omega^2) = -\omega^2.$$

THEOREM 1. *Every V-point is collinear with three W-points. The cross-ratio of these four points is equianharmonic.*

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* Presented to the Society, November 28, 1936.

† American Journal of Mathematics, vol. 53 (1931), pp. 902–910.

Next, consider any of the A_i, say $A_1(1, 0, 0, 0)$, and the triples of points

$$\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & & 1 & \omega & \omega & 1 & & 1 & \omega^2 & \omega^2 & 1 & & 1 & \omega & 1 & 1 & & 1 & 1 & \omega & 1 \\
\omega & 1 & 1 & 1 & & \omega & \omega & \omega & 1 & & \omega & \omega^2 & \omega^2 & 1 & & \omega & \omega & 1 & 1 & & \omega & 1 & \omega & 1 \\
\omega^2 & 1 & 1 & 1 & & \omega^2 & \omega & \omega & 1 & & \omega^2 & \omega^2 & \omega^2 & 1 & & \omega^2 & \omega & 1 & 1 & & \omega^2 & 1 & \omega & 1
\end{array}$$

$$\begin{array}{cccccccccccc}
1 & \omega^2 & 1 & 1 & & 1 & 1 & \omega^2 & 1 & & 1 & \omega & \omega^2 & 1 & & 1 & \omega^2 & \omega & 1 \\
\omega & \omega^2 & 1 & 1 & & \omega & 1 & \omega^2 & 1 & & \omega & \omega & \omega^2 & 1 & & \omega & \omega^2 & \omega & 1 \\
\omega^2 & \omega^2 & 1 & 1 & & \omega^2 & 1 & \omega^2 & 1 & & \omega^2 & \omega & \omega^2 & 1 & & \omega^2 & \omega^2 & \omega & 1
\end{array}$$

The points of each of these triples are collinear with A_1. Moreover the cross-ratio of A_1 and any of the triples is $(\infty, 1, \omega, \omega^2) = -\omega^2$. Taking account of the symmetry of the G_{27} we may state the following theorem.

THEOREM 2. *The 27 W-points lie 4 times on 9 lines through the A_i's, each line containing 3 of the W's. The cross-ratio of each A_i and 3 collinear W's is equianharmonic. Through every W there are 4 such lines of collinearity.*

3. *Surfaces on the G_{27}.* It is clear that

$$F = \sum C_{abcdef}(x_1^3 - x_2^3)^a(x_1^3 - x_3^3)^b(x_1^3 - x_4^3)^c(x_2^3 - x_3^3)^d \cdot (x_2^3 - x_4^3)^e(x_3^3 - x_4^3)^f X^{(g)}_{abcdef} = 0,$$

where the X's are quaternary forms of degree g, with $a+b+c+d+e+f=m$ (constant), is a surface of order $3m+g$ with the points of the G_{27} as base points. The simplest form of this kind is $C_1(x_1^3-x_2^3)+C_2(x_1^3-x_3^3)+C_3(x_1^3-x_4^3)+C_4(x_2^3-x_3^3)+C_5(x_2^3-x_4^3)+C_6(x_3^3-x_4^3)=0$. By properly relabeling the coefficients of F, this can be put into the simpler form

$$F_3 = a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^3 = 0,$$

with $a_1+a_2+a_3+a_4=0$. Among these are four cubic cones with the A_i's as vertices. For example, when $a_4=0$, and $a_1+a_2+a_3=0$, or $a_1=\alpha_2-\alpha_3$, $a_2=\alpha_3-\alpha_1$, $a_3=\alpha_1-\alpha_2$, we have the cubic cone

$$(\alpha_2 - \alpha_3)x_1^3 + (\alpha_3 - \alpha_1)x_2^3 + (\alpha_1 - \alpha_2)x_3^3 = 0,$$

which depends on one effective constant. This agrees with the fact established before that the 27 W's lie by 3 on 9 lines through A_1. The result may be stated as the following theorem.

THEOREM 3. *The cubics on the G_{27} form a linear system of dimension two, or a net. Among these are four cubic cones with the A_i's as vertices. The 9 lines through an A_i, containing the 27 W's, form an associated system of lines of a pencil of cubic cones with the common vertex A_i. The G_{27} is an associated set of base points of the net of cubics.*

Among the quartic F-surfaces may be mentioned in particular

$$F_4 = \sum(a_{ijk}x_k + a_{ijl}x_l)(x_i^3 - x_j^3) = 0,$$

in which i, j, k, l are 1, 2, 3, 4 taken in any order. F_4 depends apparently on 12 homogeneous constants, and so does

$$F_4' = (a_2x_2 + a_3x_3 + a_4x_4)x_1^3 + (b_1x_1 + b_3x_3 + b_4x_4)x_2^3 + (c_1x_1 + c_2x_2 + c_4x_4)x_3^3 + (d_1x_1 + d_2x_2 + d_3x_3)x_4^3 = 0.$$

But for F_4' to be of the form F_4, it is clear that all the coefficients of F_4' cannot be independent. In fact it is easily seen that the four identities must exist: $b_1+c_1+d_1=0$, $a_2+c_2+d_2=0$, $a_3+b_3+d_3=0$, $a_4+b_4+c_4=0$. From this follows our next theorem.

THEOREM 4. *The quartics F_4 form a linear system of dimension 7, which may be written in the form*

$$\begin{aligned}&\{(\beta_1 - \beta_3)x_2 + (\gamma_1 - \gamma_2)x_3 + (\delta_2 - \delta_3)x_4\}x_1^3\\ &+ \{(\alpha_2 - \alpha_3)x_1 + (\gamma_2 - \gamma_4)x_3 + (\delta_3 - \delta_1)x_4\}x_2^3\\ &+ \{(\alpha_3 - \alpha_4)x_1 + (\beta_3 - \beta_4)x_2 + (\delta_1 - \delta_2)x_4\}x_3^3\\ &+ \{(\alpha_4 - \alpha_2)x_1 + (\beta_4 - \beta_1)x_2 + (\gamma_4 - \gamma_1)x_3\}x_4^3 = 0.\end{aligned}$$

The tangent planes to such an F_4 at the A_i's are concurrent.

Without going into a classification and extended discussion of surfaces on the G_{27} we conclude this section by the construction of the symmetric sextic

$$F_6 = \sum(x_i^3 - x_k^3)^2 = 0, \qquad (i \neq k).$$

Denoting the elementary symmetric functions on the variables by $\phi_1=\sum x_i$, $\phi_2=\sum x_ix_k$, $\phi_3=\sum x_ix_kx_l$, $\phi_4=x_1x_2x_3x_4$, we find that F_6 has the form

$$F_6 = 3\phi_1^6 - 18\phi_1^4\phi_2 + 18\phi_1^3\phi_3 + 27\phi_1^2\phi_2^2 - 24\phi_1^2\phi_4 - 30\phi_1\phi_2\phi_3 + 24\phi_2\phi_4 - 8\phi_2^3 + 3\phi_3^2 = 0.$$

Dividing through by ϕ_1^6 and setting $x=\phi_2/\phi_1^2$, $y=\phi_3/\phi_1^3$, $z=\phi_4/\phi_1^4$, we map F_6 upon the cubic monoid

$$24z(x-1)-8x^3-30xy+27x^2+3y^2-18x+18y+3=0,$$

that is, upon a rational surface. As the mapping is birational, we have the following result.

THEOREM 5. *The sextic* $\sum(x_i^3-x_k^3)^2=0$ *is rational and has the points of the* G_{27} *as double points.*

4. *The* G_{36} *of V-Points.* In the plane of the triangle $A_1A_2A_3$ consider the syzygetic pencil of cubics $x_1^3+x_2^3+x_3^3-6\lambda x_1x_2x_3=0$, and the 12 vertices of the four flex triangles, among which are the three vertices $A_1(1,0,0)$, $A_2(0,1,0)$, $A_3(0,0,1)$. When we exclude these, there remain 9 V-points, defined by $V(\omega^\alpha,\omega^\beta,\omega^\gamma,0)$, $\alpha,\beta,\gamma\equiv 0,1,2 \pmod 3$. In a similar way we find 9 V-points in the remaining coordinate planes, hence, altogether 36 V's. A property of these points has already been stated in Theorem 1. There are 27 planes of the type

$$\omega^\alpha x_1+\omega^\beta x_2+\omega^\gamma x_3+x_4=0$$

on each of which we find 8 V's. For example, on the plane $x_1+\omega x_2+\omega^2x_3+x_4=0$ lie

$$(0,1,1,1),\quad(\omega,0,1,1),\quad(\omega,\omega,1,1),\quad(1,1,1,0),$$
$$(0,\omega,\omega^2,1),\quad(\omega^2,0,\omega^2,1),\quad(\omega^2,1,0,1),\quad(1,\omega,\omega^2,0).$$

They also lie on a conic cut out on the plane by the quadric $\omega x_1x_2+\omega^2x_1x_3+x_1x_4+\omega^2x_2x_4+\omega x_2x_4+x_2x_3=0$. A point V, say $(1,\omega,\omega^2,0)$, lies on 6 planes,

$$x_1+x_2+x_3+x_4=0,\qquad x_1+\omega x_2+\omega^2x_3+x_4=0,$$
$$\omega x_1+\omega x_2+\omega x_3+x_4=0,\qquad \omega^2x_1+\omega^2x_2+\omega^2x_3+x_4=0,$$
$$\omega x_1+\omega^2x_2+x_3+x_4=0,\qquad \omega^2x_1+x_2+\omega x_3+x_4=0.$$

THEOREM 6. *The 36 V's lie by 8 on 27 conics in 27 planes. Through each V pass 6 of these planes and one of the 36 lines through the* A_i*'s containing each 3 of the W's.*

As before we may again set up linear systems of surfaces on the G_{36} (including the A_i's), and eventually on the G_{36} and the G_{27}. We shall restrict ourselves to one particularly interesting example of a septimic which cuts the faces of the coordinate

tetrahedron outside its edges each in a *Caporali quartic, and contains both the* G_{27} *and the* G_{36}.

$$F_7 = x_1x_2x_3\{a_{14}x_1(x_2^3 - x_3^3) + a_{24}x_2(x_3^3 - x_1^3) + a_{34}x_3(x_1^3 - x_2^3)\}$$
$$+ x_1x_2x_4\{a_{13}x_1(x_2^3 - x_4^3) + a_{23}x_2(x_4^3 - x_1^3) + a_{43}x_4(x_1^3 - x_2^3)\}$$
$$+ x_1x_3x_4\{a_{12}x_1(x_3^3 - x_4^3) + a_{32}x_3(x_4^3 - x_1^3) + a_{42}x_4(x_1^3 - x_3^3)\}$$
$$+ x_2x_3x_4\{a_{21}x_2(x_3^3 - x_4^3) + a_{31}x_3(x_4^3 - x_2^3) + a_{41}x_4(x_2^3 - x_3^3)\} = 0.$$

It has the A_i*'s as triple points and the* $\overline{A_iA_k}$ *as single lines.*

UNIVERSITY OF ILLINOIS

EINSTEIN SPACES OF CLASS ONE*

BY C. B. ALLENDOERFER

1. *Introduction.* An Einstein space is defined as a Riemann space for which

$$R_{\alpha\beta} = \frac{R}{n} g_{\alpha\beta}. \tag{1}$$

We assume the first fundamental form

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta \tag{2}$$

to be non-singular, but do not restrict ourselves to the positive definite case. An $n+1$ dimensional space is said to be flat when its first fundamental form can be reduced to†

$$ds^2 = \sum_{i=1}^{n+1} c_i(dx^i)^2, \tag{3}$$

where the c_i are definitely plus one or minus one. An n dimensional Riemann space which is not flat is said to be of class one if it can be imbedded in an $n+1$ dimensional flat space. The purpose of this paper is to determine necessary and sufficient conditions that an Einstein space be of class one.

There is no problem when $n=2$, for then every space which

* Presented to the Society, September 3, 1936.

† Throughout this paper Latin indices will have the range 1 to $n+1$; Greek indices the range 1 to n.

is not flat is of class one. Neither need we consider the case $n=3$, for then every Einstein space is a space of constant curvature and consequently is of class one unless it is flat.* It has further been shown that there can exist no Einstein space with a positive definite form (2) which is of class one and for which $R=0$.† There may, however, exist spaces of class one with $R=0$ when the form (2) is indefinite. Brinkmann‡ has shown that this property is enjoyed for $n=4$ by a space with the first fundamental form

$$ds^2 = 2dxdy + 2d\phi d\theta + 2f(x, \phi)dxd\phi,$$

where $f(x, \phi)$ is any function possessing continuous first and second derivatives. In this paper we consider the case where $n>3$ and $R\neq 0$, thus leaving open the case where $R=0$ and the form (2) is indefinite. Our procedure will consist of reducing the problem to a purely algebraic one by means of a recent theorem of T. Y. Thomas;§ and then the algebraic problem is solved by straightforward elimination methods.

2. *Riemann Spaces of Class One.* Before discussing Einstein spaces, let us recall the general theory of a Riemann space of class one. Let U be a simply connected neighborhood of the n dimensional arithmetic number space. A Riemann space will be called an R_2 in U if the coefficients of the first fundamental form (2) are functions of class C^2 for $x \subset U$ and the components of the Riemann curvature tensor are of class C^1 for $x \subset U$. An R_2 is said to be imbedded in the $n+1$ dimensional flat space whose metric is given by (3) when there exist functions $y^i=\phi^i(x)$ of class C^2 in U which satisfy the mixed system

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* J. A. Schouten and D. J. Struik, *On some properties of general manifolds relating to Einstein's theory of gravitation*, American Journal of Mathematics, vol. 43 (1921), p. 214.

† E. Kasner, *The impossibility of Einstein fields immersed in flat space of five dimensions*, American Journal of Mathematics, vol. 43 (1921), p. 126. See also L. P. Eisenhart, *Riemannian Geometry*, 1926, for a discussion of Einstein spaces and for the general theory of spaces of class one on which this discussion is based.

‡ H. W. Brinkmann, *Conformal mapping of Einstein spaces*, Mathematische Annalen, vol. 94 (1925), pp. 140–141.

§ T. Y. Thomas, *Riemann spaces of class one and their characterization*, Acta Mathematica, vol. 67 (1936), pp. 169–211.

$$\frac{\partial y^i}{\partial x^\alpha} = y^i_{,\alpha},$$

$$\frac{\partial (y^i_{,\alpha})}{\partial x^\beta} = \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} y^i_{,\gamma} + e b_{\alpha\beta}\sigma^i, \tag{4}$$

$$\frac{\partial (\sigma^i)}{\partial x^\alpha} = - b_{\alpha\beta} g^{\beta\gamma} y^i_{,\gamma},$$

and

$$\sum_{i=1}^{n+1} c_i y^i_{,\alpha} y^i_{,\beta} = g_{\alpha\beta}, \qquad \sum_{i=1}^{n+1} c_i y^i_{,\alpha}\sigma^i = 0, \qquad \sum_{i=1}^{n+1} c_i \sigma^i\sigma^i = e, \tag{5}$$

where σ^i is the unit normal to R_2, $b_{\alpha\beta}$ are the coefficients of the second fundamental form of R_2, and e is definitely plus one or minus one. The functions $b_{\alpha\beta}(x)$ and σ^i are of class C^1 in U. This system is completely integrable if and only if the Gauss equations

$$R_{\alpha\beta\mu\rho} = e(b_{\alpha\mu}b_{\beta\rho} - b_{\alpha\rho}b_{\beta\mu}), \tag{6}$$

and the Codazzi equations

$$b_{\alpha\mu,\rho} = b_{\alpha\rho,\mu} \tag{7}$$

are satisfied for all $x \subset U$. Under these circumstances the system has a unique solution in U for every set of initial values $(y^i)_0$, $(y^i_{,\alpha})_0$, and $(\sigma^i)_0$ which satisfy (5) at the point P where $x = x_0$. *A given R_2 defined in a neighborhood U is of class one if and only if its curvature tensor does not vanish and there can be found a value of e and a set of functions $b_{\alpha\beta}(x)$ of class C^1 which satisfy* (6) *and* (7) *in U.*

Suppose now that we are given a Riemann space whose first fundamental form is of signature s, and the coefficients of whose second fundamental form satisfy (6) and (7) for a particular value of e. We seek to determine the signature of the flat space in which it can be imbedded. So we choose a coordinate system such that at the initial point, P, $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$, $g_{\alpha\alpha} = +1$ for $\alpha = 1, \cdots, (s+n)/2$, and $g_{\alpha\alpha} = -1$ for $\alpha = [(s+n)/2]+1, \cdots, n$. An obvious set of c_i and initial values of the variables which satisfies (5) at P is obtained by putting $c_i = +1$ for $i = 1, \cdots, (s+n)/2$, $c_i = -1$ for $i = [(s+n)/2]+1, \cdots, n$; $c_{n+1} = e$; and $(y^i_{,\alpha})_0 = \delta^i_\alpha$; $(\sigma^i)_0 = 0$ for $i = 1, \cdots, n$; $(\sigma^{n+1})_0 = 1$.

The signature of the flat space so determined is thus $s+e$. The usual argument* shows that this signature is the same for all sets of solutions of (5).

3. *Einstein Spaces of Class One.* We now pass to the consideration of Einstein spaces and suppose that we are given an Einstein space of class one which is furthermore an R_2 in U. Then there exist real functions $b_{\alpha\beta}(x)$ of class C^1 and a value of e which satisfy (6). In order to solve (6) for the $b_{\alpha\beta}(x)$, we observe that from (6) we have

$$
\begin{aligned}
&2e(b_{\gamma\tau}b_{\beta\mu}R_{\alpha\delta\sigma\rho} + b_{\gamma\rho}b_{\beta\mu}R_{\alpha\delta\tau\sigma} + b_{\gamma\sigma}b_{\beta\mu}R_{\alpha\delta\rho\tau}) \\
&\quad = R_{\alpha\beta\mu\sigma}R_{\gamma\delta\rho\tau} + R_{\alpha\beta\mu\tau}R_{\gamma\delta\sigma\rho} + R_{\alpha\beta\mu\rho}R_{\gamma\delta\tau\sigma} - R_{\delta\beta\mu\sigma}R_{\gamma\alpha\rho\tau} \\
&\quad\quad - R_{\delta\beta\mu\tau}R_{\gamma\alpha\sigma\rho} - R_{\delta\beta\mu\rho}R_{\gamma\alpha\tau\sigma} - R_{\alpha\delta\mu\tau}R_{\gamma\beta\rho\sigma} \\
&\quad\quad - R_{\alpha\delta\mu\rho}R_{\gamma\beta\sigma\tau} - R_{\alpha\delta\mu\sigma}R_{\gamma\beta\tau\rho}.
\end{aligned} \tag{8}
$$

Multiplying (8) by $g^{\alpha\sigma}g^{\delta\rho}$, summing, and using (1), we find that

$$
\begin{aligned}
&b_{\gamma\tau}b_{\beta\mu} \\
&= e\left[\frac{R}{n(2-n)}g_{\gamma\tau}g_{\beta\mu} - \frac{n}{2R(2-n)}g^{\alpha\sigma}g^{\delta\rho}(2R_{\alpha\beta\mu\rho}R_{\delta\gamma\tau\sigma} + R_{\alpha\delta\mu\tau}R_{\gamma\beta\rho\sigma})\right] \\
&= e\,D_{\gamma\tau|\beta\mu},
\end{aligned} \tag{9}
$$

where the $D_{\gamma\tau|\beta\mu}$ are thus defined. It follows at once that

$$
b_{\gamma\tau} = \pm\,(eD_{\gamma\tau|\gamma\tau})^{1/2}. \tag{10}
$$

Since $b_{\gamma\tau}$ are real, *the matrix*

$$
M \equiv \|D_{\gamma\tau|\beta\mu}\|
$$

is semi-definite of rank one in U, where $(\gamma\tau)$ indicates the row and $(\beta\mu)$ the column. We call this property condition A.

Interchanging τ and μ in (9), subtracting the result from (9), and using (6), we find that, for every point in U,

$$
\begin{aligned}
R_{\gamma\beta\tau\mu} &= \frac{R}{n(2-n)}(g_{\gamma\tau}g_{\beta\mu} - g_{\gamma\mu}g_{\beta\tau}) \\
&\quad - \frac{n}{R(2-n)}g^{\alpha\sigma}g^{\delta\rho}(R_{\alpha\beta\mu\rho}R_{\delta\gamma\tau\sigma} - R_{\alpha\gamma\mu\rho}R_{\delta\beta\tau\sigma} + R_{\alpha\delta\mu\tau}R_{\gamma\beta\rho\sigma}).
\end{aligned} \tag{11}
$$

* See, for example, Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, 1930, vol. 2, p. 26.

We call this condition B, thus having proved that conditions A and B are necessary that the given space be of class one.

To show that they are sufficient, suppose an Einstein space ($R \neq 0$) is given which satisfies A and B and which is also an R_2 in U. Since M is semi-definite of rank one in U, we know that at any point $P \subset U$ there exist a unique determination of e and real values of $b_{\alpha\beta}$ which are unique to within algebraic sign, such that (9) is satisfied. At least one of these, say $b_{\xi\eta}$, is not zero at P, and its value is given by

$$b_{\xi\eta} = \pm (eD_{\xi\eta|\xi\eta})^{1/2}, \tag{12}$$

where $D_{\xi\eta|\xi\eta} \neq 0$ at P and e is chosen such that $b_{\xi\eta}$ is real. The values of the $b_{\beta\mu}$ for other sets of indices may be obtained from

$$b_{\beta\mu} = \frac{eD_{\beta\mu|\xi\eta}}{b_{\xi\eta}}. \tag{13}$$

Since $D_{\xi\eta|\xi\eta}$ is of class C^1 in U, there exists a neighborhood $V(P)$ such that $P \subset V(P) \subset U$ within which $D_{\xi\eta|\xi\eta} \neq 0$ and $b_{\xi\eta} \neq 0$. Since the other $D_{\beta\mu|\xi\eta}$ are also of class C^1 in U, it follows that for each choice of the sign in (12), equations (12) and (13) define a constant value of e and a set of real $b_{\alpha\beta}(x)$ which are of class C^1 in $V(P)$. It is clear that the choice of sign throughout $V(P)$ is determined by its value at any point. These $b_{\alpha\beta}(x)$ and e satisfy (9) by their very definition, and since (11) is satisfied in U by hypothesis, they also satisfy (6) in $V(P)$.

Now consider a point $Q \subset V(P) \subset V(P')$, where P and P' are distinct points. By the above procedure define a definite set of $b_{\alpha\beta}(Q)$ and a value of e, say $e(Q)$, at Q, when Q is considered as an element of $V(P)$. Another set, $b'_{\alpha\beta}(Q)$ and $e'(Q)$, are defined when Q is considered as an element of $V(P')$. But since the value of e is unique at any point and is constant in any V neighborhood, it follows that e is constant throughout $V(P)+V(P')$. And since the possible values of $b_{\alpha\beta}$ at any point are unique to within algebraic sign, it is clear that the sign to be chosen in (12) for $V(P')$ can be taken such that $b_{\alpha\beta}(Q) = b'_{\alpha\beta}(Q)$. Since $b_{\alpha\beta}(x)$ and $b'_{\alpha\beta}(x)$ are of class C^1 in $V(P)$ and $V(P')$, respectively, it follows that $b_{\alpha\beta}(x) = b'_{\alpha\beta}(x)$ throughout $V(P) \cap V(P')$. Thus we have defined a solution of (6) which is of class C^1 in $V(P)+V(P')$.

Now T. Y. Thomas* has shown that this method can be used to extend this solution to the entire neighborhood U. The result is that A *and* B *are sufficient conditions for the existence of a set of* $b_{\alpha\beta}(x)$ *of class* C^1 *and a constant value of* e *which satisfy* (6) *in* U.

From this we can show at once that the $b_{\alpha\beta}(x)$ also satisfy (7) in U. For, first, the matrix $\|b_{\alpha\beta}\|$ is of rank n in U. Multiply (6) by $g^{\alpha\rho}$, sum, and make use of (1). There results

$$g_{\beta\mu}\frac{R}{n} = eg^{\alpha\rho}b_{\alpha\mu}b_{\beta\rho} - b_{\alpha\rho}g^{\alpha\rho}b_{\beta\mu}. \tag{14}$$

Now if $\|b_{\alpha\beta}\|$ were of rank $<n$ at any point $P \subset U$, the system $b_{\alpha\mu}\lambda^\mu = 0$ would have at least one non-zero solution at P. But if this were the case, (14) would show that $g_{\alpha\mu}\lambda^\mu = 0$ has at least one non-zero solution at P, but this is impossible, since we have assumed that $|g_{\alpha\beta}| \neq 0$ in U. Since T. Y. Thomas has shown in the paper mentioned above that if the rank of $\|b_{\alpha\beta}\|$ is >3, the Codazzi equations are satisfied as consequences of the Gauss equations, it follows that our $b_{\alpha\beta}(x)$ satisfy (7) in U.

THEOREM. *An Einstein space with* $n>3$ *and* $R\neq 0$ *which is an* R_2 *in a simply connected neighborhood* U *is a space of class one if and only if* $\|D_{\gamma\tau|\beta\mu}\|$ *is semi-definite of rank one in* U, *and equations* (11) *are satisfied in* U. *If* s *is the signature of the first fundamental form of an Einstein space which satisfies the above conditions, the space can be imbedded in any of the flat spaces whose first fundamental forms are of signature* $s+e$, *where* e *is determined as in* (12).

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* T. Y. Thomas, *Riemann spaces of class one and their characterization*, Acta Mathematica, vol. 67 (1936), p. 205.

AN ARITHMETIC FUNCTION

BY LEONARD CARLITZ

1. *Introduction.* The function*

$$\psi(k, m) = \sum_{st=k} \mu(s)m^t, \tag{1}$$

where $\mu(s)$ is the Möbius function, has the property

$$\psi(k, m) \equiv 0 \pmod{k}, \tag{2}$$

for arbitrary integral m. Gegenbauer† has generalized this by replacing $\mu(s)$ by an arbitrary integral-valued function $w(s)$ for which

$$\sum_{s|k} w(s) \equiv 0 \pmod{k}, \tag{3}$$

for all k. Clearly (3) holds for the function $\mu(s)$. Since (1) is equivalent to

$$\sum_{s|k} \psi(s, m) = m^k,$$

we put

$$W(k, m) = \sum_{st=k} w(s)m^t = \sum_{sde=k} w(s)\psi(e, m) = \sum_{te=k} \psi(e, m) \sum_{s|t} w(s) \tag{4}$$

and therefore by (2) and (3),

$$W(k, m) \equiv 0 \pmod{k}, \tag{5}$$

for all m. Conversely it is easy to show, by an induction on k, that (5) implies (3). Indeed, if (3) holds for all integers $<k$, it follows from (4) and (5) that

$$\psi(1, m) \sum_{s|k} w(s) = m \sum_{s|k} w(s) \equiv 0 \pmod{k}.$$

Since this must hold for all m, we may select an m prime to k, and therefore we have (3).

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* For references see Dickson's *History of the Theory of Numbers*, vol. 1, pp. 84–86. Cited as Dickson.

† See Dickson, p. 86.

2. *The Generalized Function.* In the right member of (1) replace m^t by an arbitrary integral-valued function $g(t)$, and define

$$\psi(k) = \psi(k, g) = \sum_{st=k} \mu(s)g(t). \tag{6}$$

From the definition it follows at once that

$$g(k) = \sum_{s|k} \psi(s),$$

and for $(h, k) = 1$,

$$\psi(hk) = \sum_{st=k} \mu(s)\psi(ht).$$

We shall now show that

$$\psi(k) \equiv 0 \pmod{k} \tag{7}$$

for all integers k if and only if

$$g(p^e t) \equiv g(p^{e-1}t) \pmod{p^e} \tag{8}$$

for all primes p and all integers t. Clearly we may assume in (8) that $p \nmid t$. Then if we write $k = p^e K$, where $p \nmid K$, it is easily seen that (6) implies

$$\psi(p^e k) = \sum_{st=k} \mu(s)\{g(p^e t) - g(p^{e-1}t)\}. \tag{9}$$

This shows that if (8) holds, then $\psi(k) \equiv 0 \pmod{p^e}$ for every p^e that divides k. Hence (8) is certainly a sufficient condition. Interchanging K and t in (9), and then inverting, we get

$$g(p^e K) - g(p^{e-1}k) = \sum_{s|k} \psi(p^e s),$$

from which it follows that (8) is also a necessary condition that (7) hold. Note that if (8) holds for each of two functions, it holds also for their product.

If now we replace the $\mu(s)$ of (6) by an integral-valued function $w(s)$ for which (3) is satisfied, we may define

$$W(k) = W(k, g) = \sum_{st=k} w(s)g(t)$$

as generalizing $\psi(k, g)$. Then as above,

$$W(k, g) = \sum_{st=k} \psi(s, g) \sum_{d|t} w(d),$$

and therefore if (7) and (3) hold, it follows that

$$W(k, g) \equiv 0 \pmod{k}. \tag{10}$$

Conversely it can be proved that (10) and (7) imply (3); similarly (10) and (3) imply (7).

3. *Connection with Irreducible Polynomials.* As is well known, if in (1) we put $m = p^n$, the power of a prime, the resulting function $\psi(k, p^n)$ is k times the number of irreducible polynomials of degree k in a single indeterminate, and with coefficients in the Galois field $GF(p^n)$. More generally, the number of irreducible *factorable* polynomials* in $GF(p^n)$,

$$G \equiv \prod_{j=1}^{k} (\alpha_{j0} + \alpha_{j1}x_1 + \cdots + \alpha_{js}x_s), \qquad \prod_{j=1}^{k} \alpha_{js} \neq 0,$$

is $\psi(k, p^{ns})/k$.

In the case of the general function $\psi(k) = \psi(k, g)$, for which (7) is assumed to hold, we consider a set of polynomials M with coefficients in a field (the precise nature of which need not be defined). The degree of M is assumed defined; the number of polynomials M of fixed degree m will be denoted by $f(m)$, $f(0) = 1$. It is assumed that M can be factored into a product of powers of irreducible polynomials (of the set) in essentially one way. If $\psi(k)/k$ be the number of irreducible polynomials P of degree k, we shall show that

$$mf(m) = \sum_{s=1}^{m} \psi(s) \{f(m - s) + f(m - 2s) + \cdots\}, \tag{11}$$

or what is the same thing,

$$mf(m) = \sum_{s=1}^{m} g(s) f(m - s). \tag{12}$$

We put

$$F(m) = \prod_{\deg M = m} M, \qquad \Theta(m) = \prod_{\deg P = m} P, \tag{13}$$

* Duke Mathematical Journal, vol. 2 (1936), pp. 660–670.

so that $F(m)$ is the product of all the polynomials of degree m, $\Theta(m)$ the product of the irreducible polynomials. To express $F(m)$ in terms of Θ, let

$$M = P^e A, \qquad P \nmid A,$$

where P is of degree s, say. Then by (13),

$$F(m) = \prod_{P,e} P^{e\phi_m - e\phi(P)}, \tag{14}$$

the product extending over all P, e such that $es \leqq m$, and $\phi_k(P)$ denotes the number of polynomials of degree k, not divisible by P. Evidently

$$\phi_k(P) = \begin{cases} f(k) & \text{for } k < s, \\ f(k) - f(k-s) & \text{for } k \geqq s. \end{cases}$$

Thus (14) becomes

$$F(m) = \prod_{P} P^{\sum e\phi_m - e\phi(P)};$$

the exponent in the right member is

$$\{f(m-s) - f(m-2s)\} + 2\{f(m-2s) - f(m-3s)\} + \cdots + rf(m-rs) = f(m-s) + f(m-2s) + \cdots + f(m-rs),$$

where $r = [m/s]$, the greatest integer $\leqq m/s$. Grouping together all P of equal degree, we have finally

$$F(m) = \prod_{s=1}^{m} \{\Theta(s)\}^{f(m-s)+\cdots+f(m-rs)}. \tag{15}$$

Comparison of the degree of the two members of (15) leads to

$$\begin{aligned} mf(m) &= \sum_{s=1}^{m} \psi(s) \sum_{e=1}^{r} f(m-es) \\ &= \sum_{es \leqq m} \psi(s) f(m-es) \\ &= \sum_{k \leqq m} f(m-k) \sum_{es=k} \psi(s) \\ &= \sum_{k=1}^{m} f(m-k) g(k), \end{aligned}$$

so that we have proved both (11) and (12).

4. *The L.C.M. Property.* In the paper previously referred to, the following formula appears incidentally:

$$\sum_{[s,t]=k} \psi(s, p^n)\psi(t, p^n) = \psi(k, p^{2n}), \tag{16}$$

the summation on the left extending over all s, t with least common multiple equal to k. This formula may be proved very easily; indeed it follows at once from a formula due to von Sterneck.*

Let $g_1(m)$, $g_2(m)$ denote arbitrary arithmetic functions, and $g(m) = g_1(m)g_2(m)$. Then for $\psi(k, g)$ as defined by (6), von Sterneck's formula is

$$\sum_{[s,t]=k} \psi(s, g_1)\psi(t, g_2) = \psi(k, g). \tag{17}$$

To prove this, consider the equivalent formula

$$\sum_{k|m} \sum_{[s,t]=k} \psi(s, g_1)\psi(t, g_2) = \sum_{k|m} \psi(k, g). \tag{18}$$

The summation conditions on the left are equivalent to $s \mid m$, $t \mid m$, that is, s and t independently ranging over the divisors of m. Thus we have

$$\sum_{s|m} \psi(s, g_1) \sum_{t|m} \psi(t, g_2) = g_1(m)g_2(m) = \sum_{k|m} \psi(k, g_1g_2),$$

which proves (18), and therefore (17).

If in (17) we take

$$g_1(s) = m^s, \qquad g_2(s) = n^s,$$

the formula becomes

$$\sum_{[s,t]=k} \psi(s, m)\psi(t, n) = \psi(k, mn),$$

a direct generalization of (16).

Formula (17) may be generalized to the case of m functions $g_1, \cdots, g_m$, $g = g_1g_2 \cdots g_m$,

$$\sum_{[s_1,\cdots,s_m]=k} \psi(s_1, g_1) \cdots \psi(s_m, g_m) = \psi(k, g),$$

* See Dickson, p. 151. For details of the L.C.M. calculus, see D. H. Lehmer, American Journal of Mathematics, vol. 53 (1931), pp. 843–854.

the summation extending over all sets $s_1, \cdots, s_m$, with least common multiple equal to 1.

5. *A Polynomial Analog of* $\psi(k)$. It is easy to define analogs of $\psi(k)$ having the property (2). For example, for an algebraic field, we have*

$$\psi(\mathfrak{m}, \beta) = \sum_{\mathfrak{ab}=\mathfrak{m}} \mu(\mathfrak{a})\beta^{n(\mathfrak{b})} \equiv 0 \pmod{\mathfrak{m}},$$

where $\mathfrak{m}$ is an ideal and β an integer in the field.

We now define an analog in the domain of polynomials in a single indeterminate, with coefficients in a $GF(p^n)$:

$$\psi(M, G) = \sum_{AB=M} \mu(A)G^{|B|}. \tag{19}$$

Here $\mu(A)$ is the Möbius function for the polynomial domain, and the *absolute value* $|B|$ is defined by

$$|B| = p^{nb}, \qquad b = \deg B.$$

Then it is easy to show that

$$\psi(M, G) \equiv 0 \pmod{M}, \tag{20}$$

for arbitrary polynomials G. For M irreducible, (20) reduces to Fermat's theorem.

More generally if $g(M)$ is a function of the polynomial M whose values are polynomials in $GF(p^n)$, we may define

$$\psi(M, g) = \sum_{AB=M} \mu(A)g(B),$$

and prove, exactly as above, that $\psi(M, g) \equiv 0 \pmod{M}$ if and only if

$$g(P^eM) \equiv g(P^{e-1}M) \pmod{P^e}.$$

Generally speaking, all our results for $\psi(m, g)$ carry over to the polynomial $\psi(M, g)$. In particular this is true of the L.C.M. property. The one exception is §3; there seems to be no connection between $\psi(M, G)$ and classes of irreducible polynomials.

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* Due to J. Westlund; see Dickson, p. 86.

ON LINE INTEGRALS AND DIFFERENTIAL EQUATIONS, ESPECIALLY THOSE OF DYNAMICS

BY D. C. LEWIS, JR.

1. *Introduction and Formulation of the Problem.* Recently a formula was given, exhibiting a Cartan relatively invariant line integral for a non-holonomic conservative dynamical system.* I wish to show that this formula is a special case of a more general formula which will here be developed. The formula itself is not explicitly that of a relatively invariant line integral, even in the special case previously treated. Nevertheless, under certain restrictions it can undoubtedly be put into that form.†

Let us consider the system of differential equations

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_m}{X_m}, \tag{1}$$

together with an arbitrary linear differential form $\sum_{i=1}^{m} A_i dx_i$, where the $X_i(x_1, \cdots, x_m)$ and $A_i(x_1, \cdots, x_m)$ are of class C'' in a region R. Let Γ_0 be a closed curve‡ in R, whose parametric equations are given, say, by

$$x_i = x_i^{(0)}(\tau), \quad 0 \leqq \tau \leqq 1, \quad x_i^{(0)}(0) = x_i^{(0)}(1), \quad (i = 1, \cdots, m).$$

Consider the *tube* T of trajectories of (1) which pass through Γ_0. Let Γ_{u_1} be any other similar closed curve embracing this tube; that is, Γ_{u_1} cuts each trajectory of the tube the same number of times as Γ_0 and in the same order.§ The whole of T between Γ_0 and Γ_{u_1} inclusive is assumed to be contained in R.

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* A. E. Taylor, *On the integral invariants of non-holonomic dynamical systems*, this Bulletin, vol. 40 (1934), pp. 735–742.

† For a complete treatment and bibliography of the related subject of integral invariants see E. J. Cartan's book, *Leçons sur les Invariants Intégraux*, 1922.

‡ The curve Γ_0 may have double points. For definiteness it may be assumed that the $x_i^{(0)}(\tau)$ are of class C'' except for a finite number of corners.

§ It is to be noticed that T is a *singular tube* (in a certain obvious sense) whenever Γ_0 cuts a trajectory more than once. This is necessarily the case when $m = 2$.

In its most general form the formula to which we allude gives a relation between the values of the line integral

$$\int \sum_{i=1}^{m} A_i dx_i$$

taken over the path Γ_0 and its value over the path Γ_{u_1}, the two paths being taken in proper senses.

2. *Derivation of the General Result.* T may be represented parametrically by equations of the form

$$x_i = x_i(\tau, u), \qquad x_i(0, u) \equiv x_i(1, u), \tag{2}$$

where the lines $\tau =$ constant represent the trajectories, the line $u = 0$ represents the curve Γ_0, and the line $u = u_1$ (where u_1 is any preassigned number other than zero) represents the curve Γ_{u_1}, as anticipated in the notation.* Along a trajectory on T we must have the following relations deduced from (1):

$$\frac{\partial x_i}{\partial u} = U(\tau, u) X_i[x_1(\tau, u), \cdots, x_m(\tau, u)], \quad (i = 1, 2, \cdots, m). \tag{3}$$

The factor of proportionality U is at most a function of τ and u. For constant u equations (2) represent a closed curve Γ_u. Let

$$J(u) = \int_{\Gamma_u} \sum_{i=1}^{m} A_i dx_i = \int_0^1 \left(\sum_{i=1}^{m} A_i \frac{\partial x_i}{\partial \tau} \right) d\tau.$$

We find by a straightforward process of differentiation, integration by parts, and substitution from (3), that

$$\frac{dJ}{du} = J'(u) = \int_0^1 \left(\sum_{i=1}^{m} U \beta_i \frac{\partial x_i}{\partial \tau} \right) d\tau, \tag{4}$$

where

$$\beta_i = \sum_{j=1}^{m} \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) X_j. \tag{5}$$

Thus we get the required relation connecting $J(u_1)$ and $J(0)$:

* This can obviously be done in an infinite number of ways, such that $x_i(\tau, u)$ is of class C'' except on a finite number of trajectories corresponding to the corners of Γ_0, where $\partial x_i/\partial u$ is continuous but $\partial x_i/\partial \tau$ and the second partial derivatives may not be.

$$(6) \qquad J(u_1) = J(0) + \int_0^1 d\tau \int_0^{u_1} \left(\sum_{i=1}^{m} U\beta_i \frac{\partial x_i}{\partial \tau} \right) du.$$

In case R is a small neighborhood of a point where not all the X's vanish simultaneously, it is possible to write the iterated integral on the right as a line integral about Γ_{u_1}. This representation is rather artificial, however, and we shall not insist upon it.* Furthermore the existence of a relatively invariant line integral in this case can yield no invariantive properties of the system (1). It is hardly more than a reflection of the fact that the system (1) can be transformed in such a region into the form $dx_1/0 = dx_2/0 = \cdots = dx_{m-1}/0 = dx_m/1$.

3. *An Important Special Case.* We now consider the special case when $X_m \equiv 1$. In this case it is usual to write t in place of x_m. From (3) and (5), we see that

$$(7) \qquad \begin{cases} \dfrac{\partial t}{\partial u} = U(\tau, u), \\ \beta_i = \displaystyle\sum_{j=1}^{m-1} \left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right) X_j + \frac{\partial A_i}{\partial t} - \frac{\partial A_m}{\partial x_i}, \\ \qquad\qquad (i = 1, 2, \cdots, m-1), \\ \beta_m = \displaystyle\sum_{j=1}^{m-1} \left(\frac{\partial A_m}{\partial x_j} - \frac{\partial A_j}{\partial t} \right) X_j. \end{cases}$$

4. *The Application to Dynamics.* Let us apply the special case of the preceding paragraph to the equations of motion of a dynamical system (not necessarily conservative), written in the following form:

$$(8) \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} + Q_i, \quad (i = 1, 2, \cdots, n),$$

taking $q_i = x_{2i-1}$, $p_i = x_{2i}$, $X_{2i-1} = \partial H/\partial p_i$, $X_{2i} = -(\partial H/\partial q_i) + Q_i$, $(2n = m-1)$, where H and the Q's are functions of the q's, p's, and t. The linear differential form which we consider along with (8) is $\sum_{i=1}^{n} p_i dq_i - H dt$, so that

$$A_{2i-1} = x_{2i}, \quad A_{2i} = 0, \quad A_m = A_{2n+1} = -H.$$

* Taylor analogously makes the tacit assumption that the left hand side of his equation (15) is a line integral; loc. cit., p. 740.

A straightforward calculation based on (7) shows us that

$$\beta_{2i-1} = Q_i, \qquad \beta_{2i} = 0, \qquad \beta_m = \beta_{2n+1} = -\sum_{i=1}^{n} \frac{\partial H}{\partial p_i} Q_i.$$

Hence, if

$$J(u) = \int_{\Gamma_u} \sum_{i=1}^{n} p_i dq_i - H dt,$$

we see from (4) that

$$(9) \quad J'(u) = \int_0^1 \left[\sum_{i=1}^{n} Q_i \frac{\partial q_i}{\partial \tau} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} Q_i \frac{\partial t}{\partial \tau} \right] \frac{\partial t}{\partial u} d\tau,$$

and thus

$$(10) \quad J(u_1) = J(0) + \int_0^1 d\tau \int_0^{u_1} \left[\sum_{i=1}^{n} Q_i \frac{\partial q_i}{\partial \tau} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} Q_i \frac{\partial t}{\partial \tau} \right] \frac{\partial t}{\partial u} du.$$

This is a generalization of Taylor's formula for non-holonomic systems, as we shall show in §5.

Another interesting result is obtained as a special case by supposing Γ_0 and Γ_{u_1} to lie in planes $t = \text{const}$. In this case one may take $t = u + t_0$ (where t_0 is a constant) and hence $\partial t/\partial u = 1$ and $\partial t/\partial \tau = 0$. The curve Γ_u may be thought of as a closed curve Γ_t' in the phase space steadily moving with the time. It is to be noted that, for a fixed t, Γ_t' may be preassigned instead of Γ_{t_0}' as previously. An immediate corollary of (9) informs us that *the time rate of change of the line integral*

$$\int_{\Gamma_t'} \sum_{i=1}^{n} p_i dq_i$$

is equal to the line integral

$$\int_{\Gamma_t'} \sum_{i=1}^{n} Q_i dq_i.$$

This result has the advantage of being expressible without the explicit introduction of the parameters τ and u. A result of this type can also be stated for the system of §3.

5. *Non-Holonomic Dynamical Systems.* We next consider a conservative non-holonomic system subject to the constraints

$\sum_{i=1}^{n} a_{\alpha i}\delta q_i + a_\alpha \delta t = 0$, $(\alpha = 1, \cdots, k < n)$, in which the a's are functions of the q's and t. We may write the equations of motion in the form (8), if we take

$$Q_i = \sum_{\alpha=1}^{k} a_{\alpha i}\lambda_\alpha, \tag{11}$$

where the λ's are functions of the p's, q's, and t, chosen in such a manner that the equations (8) admit the following k first integrals:

$$\sum_{i=1}^{n} a_{\alpha i}\frac{\partial H}{\partial p_i} + a_\alpha = c_\alpha, \text{ a constant}, \quad (\alpha = 1, \cdots, k). \tag{12}$$*

Thus, since $\partial H/\partial p_i = dq_i/dt$, a motion which initially satisfies the conditions of constraint will continue to do so. The motions of the non-holonomic system are precisely those motions of (8) for which $c_1 = c_2 = \cdots = c_k = 0$. In considering the motions for which all these constants are *not* necessarily zero, we shall, however, be considering also certain motions which are not kinematically possible, that is, which do not satisfy the conditions of constraint. If we multiply (12) by λ_α and sum over α, we obtain

$$\sum_{i,\alpha} a_{\alpha i}\lambda_\alpha \frac{\partial H}{\partial p_i} + \sum_{\alpha} a_\alpha \lambda_\alpha = \sum_{\alpha} c_\alpha \lambda_\alpha.$$

But this may be written with the help of (11) in the form

$$\sum_{i=1}^{n} Q_i \frac{\partial H}{\partial p_i} = \sum_{\alpha=1}^{k} (c_\alpha - a_\alpha)\lambda_\alpha. \tag{13}$$

Substituting from (11) and (13) into (10), we find that

* The λ's may be calculated as follows: Differentiate (12) totally with respect to t and substitute for the derivatives of the p's and q's their values as given by (8). In virtue of (11) the result is a system of k linear equations in the unknowns $\lambda_1, \cdots, \lambda_k$. In the most important case where the kinetic energy can be written as a positive definite form in the velocities, the determinant of this system can be shown to be different from zero, the k equations of constraint being linearly independent. The determinant can even be written in diagonal form by a preliminary normalization of the equations of constraint. See G. Prange, *Die Allgemeinen Integrationsmethoden der Analytischen Mechanik*, Encyklopädie der Mathematischen Wissenschaften, IV Mechanik, 2 (1935), p. 558.

$$J(u_1) = J(0) + \int_0^1 d\tau \int_0^{u_1} \left[\sum_{i=1}^{n} \left(\sum_{\alpha=1}^{k} a_{\alpha i} \lambda_\alpha \right) \frac{\partial q_i}{\partial \tau} - \left(\sum_{\alpha=1} (c_\alpha - a_\alpha) \lambda_\alpha \right) \frac{\partial t}{\partial \tau} \right] \frac{\partial t}{\partial u} du.$$

Since c's may have different values along different trajectories of T, they are here to be considered as functions of τ. If we take all the c's equal to zero, thus considering only the actual motions of the non-holonomic system, we have exactly the result of Taylor, save for a difference in notation.*

6. *Characterization of Differential Equations by Line Integrals.* Going back to the general case of §§1 and 2, we now state in conclusion that *the validity of* (4), *or* (6), *for all tubes T characterizes completely the differential equations* (1), *provided that the m-rowed determinant,*

$$\left| \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right|, \qquad (i, j = 1, \cdots, m),$$

is not zero. For, suppose the A's and β's are given such that (4) holds for all tubes T. Then, $U(\partial x_i/\partial \tau)$ may be taken almost arbitrarily, and the equations (5) result. But these are sufficient to determine the X's in terms of the A's and β's. The details are left to the reader.

In the more restricted case of §3, it is only necessary that the $(m-1)$-rowed determinant,

$$\left| \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \right|, \quad (i, j = 1, \cdots, m-1),$$

be distinct from zero. For, in this case, X_m is given $\equiv 1$ and equations (7) are surely sufficient to determine the other X's.

This latter condition is fulfilled in the case of the dynamical equations and the *action* line integral of §4.

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* Loc. cit., equation (15). Taylor's λ's are the negatives of ours. He also takes the curve Γ_0 (though not Γ_{u_1}) in a plane $t = \text{const.}$

A SIMPLIFIED SOLUTION OF THE EQUATION $\Delta y(x) = F(x)$*

BY I. M. SHEFFER

We have elsewhere considered, from the point of view of a local solution, the equation

$$\Delta y(x) \equiv y(x+1) - y(x) = F(x), \tag{1}$$

and certain related equations.† It has now been found possible to simplify the form of the solution of (1) by altering the method. This we propose to show in the present note. Suitably modified, this method may well be expected to apply to other equations.

Let Δ be the difference operator. If t is a parameter, then

$$\Delta[e^{tx}] = e^{tx}(e^t - 1), \tag{2}$$

so that

$$e^{tx}(e^t - 1) = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{3}$$

where $P_n(x)$ is the polynomial defined by

$$P_n(x) = \Delta\left[\frac{x^n}{n!}\right] = \frac{1}{n!}\{(x+1)^n - x^n\}. \tag{4}$$

Suppose that the coefficient of t^n is multiplied (for every n) by $n!$ on both sides of (3). This yields

$$\frac{1}{1-t(x+1)} - \frac{1}{1-tx} = \sum_{0}^{\infty} n!P_n(x)t^n. \tag{5}$$

We now transform (5) by replacing t by $1/t$ and dividing through by t:

$$\frac{1}{t-1-x} - \frac{1}{t-x} = \sum_{0}^{\infty} \frac{n!P_n(x)}{t^{n+1}}. \tag{6}$$

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* Presented to the Society, September 5, 1936.

† *A local solution of the difference equation $\Delta y(x) = F(x)$ and of related equations*, Transactions of this Society, vol. 39 (1936), pp. 345–379.

Our aim is to obtain a P_n-expansion for the single term $1/(t-x)$. This requires that we get rid of one of the terms on the left side of (6), a condition that can be realized in two ways by iteration. This will give us two expansions for $1/(t-x)$, both of which we need.

Let t in (6) be replaced successively by $t-1, t-2, \cdots, t-k$, and the corresponding relations (including (6)) added. This gives

$$(7)\qquad \frac{1}{t-(k+1)-x}-\frac{1}{t-x} = \sum_{n=0}^{\infty} n!P_n(x)\left[\frac{1}{t^{n+1}}+\frac{1}{(t-1)^{n+1}}+\cdots+\frac{1}{(t-k)^{n+1}}\right].$$

Likewise, on replacing t in (6) by $t+1, t+2, \cdots, t+k$ and adding (this time not using (6)), we obtain

$$(8)\qquad \frac{1}{t-x}-\frac{1}{t+k-x} = \sum_{0}^{\infty} n!P_n(x)\left[\frac{1}{(t+1)^{n+1}}+\cdots+\frac{1}{(t+k)^{n+1}}\right].$$

If k is allowed to become infinite, the suggested (but as yet *formal*) relations are

$$(9)\qquad \frac{1}{t-x}\sim -\sum_{n=0}^{\infty} P_n(x)\left\{\frac{1}{t^{n+1}}+\frac{1}{(t-1)^{n+1}}+\cdots\right\}n!,$$

$$(10)\qquad \frac{1}{t-x}\sim \sum_{n=0}^{\infty} P_n(x)\left\{\frac{1}{(t+1)^{n+1}}+\frac{1}{(t+2)^{n+1}}+\cdots\right\}n!.$$

The series in the braces converge for $n>0$, but diverge for $n=0$. Fortunately $P_0(x)\equiv 0$; hence the series can start at $n=1$. We can determine the region of convergence of (9) and (10) as follows.

Define functions* $H(t)$ and $K(t)$ by

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* The additional terms $-1/r, 1/r$ have been inserted in order to secure convergence. The functions H, K are well known in the theory of the Gamma Function, and it is of interest to observe in how natural a manner they arise in the present paper.

$$(11)\quad H(t)=\sum_{r=1}^{\infty}\left[\frac{1}{t+r}-\frac{1}{r}\right],\quad K(t)=\frac{1}{t}+\sum_{r=1}^{\infty}\left[\frac{1}{t-r}+\frac{1}{r}\right].$$

These series converge uniformly in any bounded region whose distance (respectively) from the set of points $-1, -2, -3, \cdots$; $0, 1, 2, \cdots$, is positive. Now

$$(12)\quad \frac{d^nH(t)}{dt^n}=(-1)^n\cdot n!\sum_{r=1}^{\infty}\frac{1}{(t+r)^{n+1}},$$
$$\frac{d^nK(t)}{dt^n}=(-1)^n\cdot n!\sum_{r=0}^{\infty}\frac{1}{(t-r)^{n+1}}.$$

Equations (9) and (10) can therefore be reformulated as

$$(13)\quad \frac{1}{t-x}=-\sum_{0}^{\infty}(-1)^n\cdot K^{(n)}(t)P_n(x),$$

$$(14)\quad \frac{1}{t-x}=\sum_{0}^{\infty}(-1)^nH^{(n)}(t)P_n(x).$$

We now determine if, and for what values, (13) and (14) are valid. Let $M(x, t)$ denote the right hand member of (13). From

$$(a)\quad K(u)=\sum_{0}^{\infty}\frac{(u-t)^n}{n!}K^{(n)}(t),$$

we obtain (using (4)),

$$(b)\quad M(x,t)=K(t-x)-K(t-x-1),$$

and therefore from (11),

$$(c)\quad M(x,t)=\sum_{r=0}^{\infty}\left\{\left[\frac{1}{t-x-r}-\frac{1}{r}\right]-\left[\frac{1}{t-x-1-r}-\frac{1}{r}\right]\right\}$$
$$=\frac{1}{t-x}.$$

That is, (13) is valid whenever the series converges. Now (a) holds provided $t\neq 0, 1, 2, \cdots$, and

$$|u-t|<\min\{|t|, |t-1|, |t-2|, \cdots\}.$$

For (b) we therefore require that*

(i) $|x|$ and $|x+1|<\min\{|t|, |t-1|, |t-2|, \cdots\}$.

* Condition (i) implies the previous condition $t\neq 0. 1, 2, \cdots$.

It follows from (i) that $t-x \neq 0, 1, 2, \cdots$, so that (c) is valid.

To sum up: *Expansion* (13) *is valid for all* x *and* t *satisfying condition* (i); and series (13) converges uniformly for x and t in any regions for which (i) holds *uniformly*. (That is, for all x and t in their respective regions, there is a positive number λ such that $\min\{|t|, |t-1|, \cdots\} \geqq \lambda$, and a positive number $\sigma < \lambda$ such that $|x|$ and $|x+1|$ never exceed σ.)

In like manner it can be shown that *expansion* (14) *is valid for all* x *and* t *satisfying the condition*

$$\text{(ii)} \quad |x| \text{ and } |x+1| < \min\{|t+1|, |t+2|, |t+3|, \cdots\};$$

and (14) converges uniformly for x and t in any regions for which (ii) holds *uniformly*.

With points 0 and -1 as centers, draw two arcs of radius σ (any number exceeding $1/2$). Let γ_1 denote the right hand arc and γ_2 the left. They meet on the line $R(x) = -1/2$. Let $\mathcal{L}$ be the open region enclosed by γ_1 and γ_2. Now let $\mathcal{E}$ be any closed set in $\mathcal{L}$, and restrict x to lie in $\mathcal{E}$. It is readily seen that if t traverses arc γ_2, condition (i) is satisfied uniformly, so that (13) holds uniformly. Similarly, if t traverses γ_1, series (14) converges uniformly.

Let $F(x)$ be any function analytic at $x = -1/2$. We can find a number $\sigma > 1/2$ so that if the arcs γ_1, γ_2 are drawn with radius σ, then $F(x)$ is analytic in $\mathcal{L}$ and on its boundary $C = \gamma_1 + \gamma_2$. If, then, we substitute for $1/(t-x)$ in Cauchy's integral

$$F(x) = \frac{1}{2\pi i} \int_C \frac{F(t)}{(t-x)}\, dt$$

the expansions (13) and (14), we obtain the following theorem.

THEOREM 1. *The function* $F(x)$, *analytic about* $x = -1/2$, *has the* P_n*-expansion*

$$F(x) = \sum_{n=0}^{\infty} f_n P_n(x), \tag{15}$$

where

$$f_n = \frac{(-1)^n}{2\pi i} \int_{\gamma_1} F(t) H^{(n)}(t)\, dt - \frac{(-1)^n}{2\pi i} \int_{\gamma_2} F(t) K^{(n)}(t)\, dt, \tag{16}$$

the integrals taken so that the boundary C is traversed in the positive sense. Series (15) *converges uniformly for x in every closed region in* $\mathcal{L}$, *and is therefore valid for all x in* $\mathcal{L}$.

From this we obtain the following result.

THEOREM 2.* *The series*

$$y(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!} \tag{17}$$

converges in a circle of radius exceeding 1/2, *and in some neighborhood of* $x = -1/2$ *the function* $y(x)$ *is a solution of the equation*

$$\Delta y(x) = F(x). \tag{1}$$

PENNSYLVANIA STATE COLLEGE

ON THE SUMMABILITY BY POSITIVE TYPICAL MEANS OF SEQUENCES $\{f(n\theta)\}$†

BY M. S. ROBERTSON

1. *Introduction.* In a recent paper‡ the author required an inequality for the expression

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\sin k\theta| \leqq \frac{1}{\pi} \int_0^{\pi} |\sin\theta| \, d\theta = \frac{2}{\pi}, \tag{1}$$

which apparently is due to T. Gronwall.§ This inequality suggests immediately the question: For what functions $f(\theta)$, defined in the interval $(-\pi, \pi)$, are we permitted to write

$$F(\theta; f) \equiv \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(k\theta) \leqq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta? \tag{2}$$

More generally, we may ask: For what functions $f(\theta)$ and sequences $\{a_n\}$ of positive numbers is the following true:

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* See Transactions of this Society, loc. cit., p. 359.

† Presented to the Society, April 11, 1936.

‡ See M. S. Robertson, *On the coefficients of a typically-real function*, this Bulletin, vol. 41 (1935), p. 569.

§ See Transactions of this Society, vol. 13 (1912), pp. 445–468.

$$(3)\qquad M(\theta;f)\equiv\lim_{n\to\infty}\sum_{k=1}^{n}a_kf(k\theta)\cdot\left(\sum_{1}^{n}a_k\right)^{-1}\leqq\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)d\theta?$$

It is the purpose of this paper to answer these questions in part by giving sufficiency conditions when the function $f(\theta)$ possesses a uniformly converging Fourier series. It will be evident from the discussion to follow that the inequality (2) is not true for all integrable functions $f(\theta)$.

2. *An Expansion Formula for* $M(\theta;f)$. We shall adopt the following notation. Let $f(\theta)$ be an integrable function in the sense of Lebesgue, defined over the interval $-\pi\leqq\theta\leqq\pi$, and periodic outside of this interval, whose Fourier series

$$(4)\qquad f(\theta)\sim c_0+\sum_{1}^{\infty}(b_m\sin m\theta+c_m\cos m\theta)$$

converges uniformly in the closed interval $-\pi\leqq\theta\leqq\pi$. Let $\{a_n\}$ be any sequence of non-negative real numbers. Let

$$(5)\qquad \overline{P}(\theta)\equiv\overline{\lim_{n\to\infty}}\,P_n(\theta)\equiv\overline{\lim_{n\to\infty}}\sum_{k=1}^{n}a_k\cos k\theta\cdot\left(\sum_{1}^{n}a_k\right)^{-1},$$

$$(6)\qquad \overline{Q}(\theta)\equiv\overline{\lim_{n\to\infty}}\,Q_n(\theta)=\overline{\lim_{n\to\infty}}\sum_{k=1}^{n}a_k\sin k\theta\cdot\left(\sum_{1}^{n}a_k\right)^{-1}.$$

We may denote by $\underline{P}(\theta)$ and $\underline{Q}(\theta)$ the corresponding functions obtained by taking inferior limits. If $\overline{P}(\theta)\equiv\underline{P}(\theta)$, we denote each simply by $P(\theta)$. A similar remark holds for $Q(\theta)$. Let

$$(7)\qquad \overline{M}(\theta;f)\equiv\overline{\lim_{n\to\infty}}\,M_n(\theta)\equiv\overline{\lim_{n\to\infty}}\sum_{k=1}^{n}a_kf(k\theta)\cdot\left(\sum_{1}^{n}a_k\right)^{-1},$$

$$(8)\qquad \underline{M}(\theta;f)=\underline{\lim_{n\to\infty}}\,M_n(\theta).$$

With this notation, we obtain an infinite series for $\overline{M}(\theta;f)$:

$$M_n(\theta)\cdot\sum_{1}^{n}a_k=c_0\cdot\sum_{1}^{n}a_k+\sum_{k=1}^{n}a_k\left(\sum_{m=1}^{\infty}b_m\sin mk\theta+c_m\cos mk\theta\right)$$

$$=c_0\cdot\sum_{1}^{n}a_k+\sum_{m=1}^{\infty}\left[b_m\left(\sum_{k=1}^{n}a_k\sin mk\theta\right)+c_m\left(\sum_{k=1}^{n}a_k\cos mk\theta\right)\right],$$

$$(9) \qquad M_n(\theta) = c_0 + \sum_{m=1}^{\infty} (b_m Q_n(m\theta) + c_m P_n(m\theta)).$$

Since the Fourier series (4) converges uniformly for $-\pi \leqq \theta \leqq \pi$, given $\epsilon > 0$, we can choose $N(\epsilon)$ sufficiently large so that, for all θ and k,

$$\left| \sum_{m=N+1}^{\infty} (b_m \sin mk\theta + c_m \cos mk\theta) \right| < \epsilon.$$

Hence

$$\sum_{k=1}^{n} a_k \cdot \left| \sum_{m=N+1}^{\infty} (b_m Q_n(m\theta) + c_m P_n(m\theta)) \right|$$

$$= \left| \sum_{m=N+1}^{\infty} \left[b_m \left(\sum_{k=1}^{n} a_k \sin mk\theta \right) + c_m \left(\sum_{k=1}^{n} a_k \cos k\theta \right) \right] \right|$$

$$= \left| \sum_{k=1}^{n} a_k \left(\sum_{N+1}^{\infty} (b_m \sin mk\theta + c_m \cos mk\theta) \right) \right| < \epsilon \cdot \sum_{1}^{n} a_k.$$

Consequently we have, for all θ and n,

$$(10) \qquad \left| \sum_{m=N+1}^{\infty} (b_m Q_n(m\theta) + c_m P_n(m\theta)) \right| < \epsilon.$$

On account of (10), we obtain from (9) in passing to the limit,

$$(11) \qquad \overline{M}(\theta; f) = c_0 + \sum_{m=1}^{\infty} (b_m \overline{Q}(m\theta) + c_m \overline{P}(m\theta)),$$

$$(12) \qquad \underline{M}(\theta; f) = c_0 + \sum_{m=1}^{\infty} (b_m \underline{Q}(m\theta) + c_m \underline{P}(m\theta)).$$

These two series are uniformly convergent for all θ since (10) holds for all n. If it should happen that

$$\overline{Q}(\theta) \equiv \underline{Q}(\theta) \equiv Q(\theta), \qquad \overline{P}(\theta) \equiv \underline{P}(\theta) \equiv P(\theta),$$

then the limit in (3) will exist.

If we make the substitution in (11) and (12) for the Fourier coefficients, $(m \geqq 1)$,

$$(13) \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \sin m\phi d\phi, \qquad c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \cos m\phi d\phi,$$

and invert the order of integration and summation which is permissible since the series is uniformly convergent in θ, we find

$$(14) \qquad \overline{M}(\theta; f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\bar{g}(\theta, \phi)d\phi,$$

$$(15) \qquad \underline{M}(\theta; f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\phi)\underline{g}(\theta, \phi)d\phi,$$

where

$$(16) \qquad \bar{g}(\theta, \phi) \equiv \frac{1}{2} + \sum_{m=1}^{\infty} (\overline{Q}(m\theta)\cdot \sin m\phi + \overline{P}(m\theta)\cdot \cos m\phi)$$

is uniformly convergent in θ and ϕ, and where $\underline{g}(\theta, \phi)$ has the corresponding definition in terms of $\underline{Q}(\theta)$ and $\underline{P}(\theta)$. If $\bar{g}(\theta, \phi) \equiv \underline{g}(\theta_1, \phi) \equiv g(\theta, \phi)$ for all θ and ϕ, we can be sure that $M(\theta; f)$ in (3) exists.

If $\overline{Q}(\theta) \equiv \underline{Q}(\theta) \equiv Q(\theta)$, $\overline{P}(\theta) \equiv \underline{P}(\theta) \equiv P(\theta)$, then it is seen from (11) and (12) that the necessary and sufficient condition for the inequality expressed in (3) is that the function

$$(17) \qquad \phi(\theta) = \sum_{m=1}^{\infty} (b_m Q(m\theta) + c_m P(m\theta))$$

should be non-positive for all values of θ.

3. *The Functions $P(\theta)$ and $Q(\theta)$ for Special Sequences $\{a_n\}$.* Let us now consider a more restricted class of sequences $\{a_n\}$ satisfying the following conditions:

(18a) The sequence $\{a_n\}$ of non-negative numbers a_n is to be composed of two subsequences $\{a_{2n+1}\}$ and $\{a_{2n}\}$ each of which is monotone (either non-increasing or non-decreasing).

$$(18b) \qquad \mu_n \equiv \sum_{1}^{n} a_k \text{ diverges to } +\infty \text{ with } \lim_{n\to\infty} \mu_n/\mu_{n-1} = 1.$$

$$(18c) \qquad \lim_{n\to\infty} \mu_n^{-1}\cdot \sum_{k=1}^{n} (-1)^k a_k = a, \qquad (-1 \leqq a \leqq 1).$$

With sequences $\{a_n\}$ of this latter type we can find $P(\theta)$ and $Q(\theta)$. For $\theta \neq k\pi$, (k an integer), we have the identity

$$(19)\qquad \sum_{k=1}^{n} a_k \sin k\theta = (2\sin\theta)^{-1} \cdot \left\{ \sum_{k=1}^{n} (a_{k+1} - a_{k-1}) \cos k\theta + a_1 - a_{n+1}\cos n\theta - a_n \cos (n+1)\theta \right\}.$$

Hence

$$\overline{\lim_{n\to\infty}} \left| Q_n(\theta) \right| \leqq (2\sin\theta)^{-1} \left\{ \overline{\lim_{n\to\infty}} \mu_n^{-1} \left(\sum_{k=1}^{n} \left| a_{k+1} - a_{k-1} \right| + a_1 + a_{n+1} + a_n \right) \right\} \leqq (2\sin\theta)^{-1} \left\{ \overline{\lim_{n\to\infty}} \frac{2(a_1 + a_2 + a_n + a_{n+1})}{\mu_n} \right\} = 0$$

by (18a) and (18b). It follows that

$$\overline{Q}(\theta) \equiv \underline{Q}(\theta) \equiv Q(\theta) \equiv 0.$$

Again, for $\theta \neq k\pi$, we have the identity

$$(20)\qquad \sum_{k=1}^{n} a_k \cos k\theta = (2\sin\theta)^{-1} \cdot \left\{ \sum_{k=1}^{n} (a_{k+1} - a_{k-1}) \sin k\theta + a_1 - a_{n+1}\sin n\theta - a_n \sin (n+1)\theta \right\},$$

from which we obtain

$$\overline{P}(\theta) \equiv \underline{P}(\theta) \equiv P(\theta) \equiv 0$$

for $\theta \neq k\pi$. If $\theta = 2p\pi$, (p an integer), evidently $P(\theta) = 1$. If $\theta = (2p+1)\pi$, $P(\theta) = a$ on account of (18c).

Substituting these values for $P(\theta)$ and $Q(\theta)$ in (11) or (12), we have the following values for $M(\theta; f)$ of (3):

$$(21a)\qquad \frac{\theta}{\pi} \text{ irrational}, \quad M(\theta; f) = c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) d\theta.$$

(21b) $\dfrac{\theta}{\pi} = \dfrac{2r}{s}$, where r and s are integers, $(2r, s) = 1$,

$$M\left(\frac{2r\pi}{s}; f\right) = c_0 + \sum_{m=1}^{\infty} c_{ms}$$

$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left\{ \frac{\sin (2n+1)\dfrac{s\theta}{2}}{\sin \dfrac{s\theta}{2}} \right\} d\theta.$$

(21c) $\dfrac{\theta}{\pi} = \dfrac{2r+1}{s}$, $(2r+1, s) = 1$,

$$M\left(\frac{2r+1}{s}\pi; f\right) = c_0 + a\sum_{m=1}^{\infty} c_{(2m-1)s} + \sum_{m=1}^{\infty} c_{2ms}$$

$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \left\{ \frac{\sin (2n+1)s\theta + 2a \sin ns\theta \cdot \cos ns\theta}{\sin s\theta} \right\} d\theta.$$

Hence for sequences of the type (18) the necessary and sufficient condition for the inequality (3) is that

(22)
$$\sum_{m=1}^{\infty} c_{ms} \leqq 0 \qquad \text{for all odd positive integers } s, \text{ and}$$

$$\sum_{m=1}^{\infty} c_{2ms} + a\sum_{m=1}^{\infty} c_{(2m-1)s} \leqq 0 \qquad \text{for all positive integers } s.$$

A sufficient condition for (22) is $c_m \leqq 0$ for $m = 1, 2, 3, \cdots$. In particular if $f(\theta)$ is an even function convex for $0 \leqq \theta \leqq \pi$, it is well known that its Fourier series is of the form

$$f(\theta) \sim c_0 + \sum_{1}^{\infty} c_{2m} \cos 2m\theta,$$

where $c_{2m} \leqq 0$, and where the series converges uniformly in the closed interval $0 \leqq \theta \leqq \pi$. Hence for sequences of type (18) and any even convex function $f(\theta)$, (3) and (1) are true. In particular, if $f(\theta) = |\sin \theta|$, we obtain (1).

YALE UNIVERSITY

POSTULATES FOR BOOLEAN ALGEBRA IN TERMS OF TERNARY REJECTION*

BY ALBERT WHITEMAN†

1. *Introduction.* The operation of ternary rejection‡ in Boolean algebra is the operation ( ) given by $(abc)=a'b'+b'c'+c'a'$. In this paper, I shall present a set of postulates for Boolean algebras in which ternary rejection is taken as the only primitive idea, besides that of *class*. As a result, all the special Boolean elements are introduced with an elegance not possible in any other set known to the author. Thus, the *negative* of an element is defined in terms of the primitive ideas, and then *any* two contradictory elements are chosen to represent the *zero* element and the *universe* element of the resulting Boolean algebra.

We prove the *sufficiency* of the new postulates for Boolean algebra by deriving from them the well known Whitehead-Huntington set;§ the proof of *necessariness* consists in the converse derivation. Finally, we establish the *consistency* and *independence* of the postulates by furnishing proof-systems of the usual type.

2. *The New Postulates.* The new postulates have as undefined ideas a *class* K and a *ternary operation* ( ). The postulates are the propositions A_1-A_5 below. In Postulates A_3-A_5 the condition *if the elements involved and their indicated combinations belong to* K is to be understood.

POSTULATE A_1. *K contains at least two distinct elements.*

POSTULATE A_2. *If a, b, c are elements of K, (abc) is an element of K.*

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‡ For other papers dealing with ternary operations see A. B. Kempe, *On the relation between the logical theory of classes and the geometrical theory of points*, Proceedings of the London Mathematical Society, (1), vol. 21 (1890), pp. 147–182; Orrin Frink, *The operations of Boolean algebras*, Annals of Mathematics, (2), vol. 27 (1925–1926), pp. 477–490; see also the bibliography at the end of Frink's paper.

§ See the Transactions of this Society, vol. 5 (1904), pp. 288–309.

POSTULATE A_3. $(abc) = (bca)$.

DEFINITION 1. $a' = (aaa)$.

POSTULATE A_4. $(a'bb') = a$.

POSTULATE A_5.* $[ab(cde)'] = [(abc)'(abd)'e]$.

3. *Theorems.* We now list a number of theorems of the resulting algebra.

1. $a'' = a$, *where* $a'' = (a')'$.
2. $(aab) = a'$.
3. $(abc) = (acb)$.
4. $(abc) = (cba) = (bca) = (acb) = (cab) = (bac)$.
5. $[a'(abc)'(a'b'c')'] = a$.
6. $[a(abc)'(ab'c')'] = a'$.
7. $(abc) = [(abd)'(abd')'c]$.
8. $[d'(abc)'(a'b'c')'] = d$.
9. *If* $(a'bc) = a$ *for all* a, *then* $c = b'$.
10. $(abc)' = (a'b'c')$.

4. *Proofs of the Preceding Theorems.*

PROOF OF 1. $a'' = (a'''a'a'') = (a'a''a''') = a$, by A_4, A_3, A_4.

PROOF OF 2. $(aab) = [(abb')'(abb')'b] = [ab(b'b'b)'] = (abb') = a'$, by A_4, 1, A_5, A_4, 1, A_4, 1.

PROOF OF 3.† Put $(acb)' = d$, $[c(abc)'b]' = e$. Then

$(abc) = [ab(ccd)'] = [ab(cdc)'] = [(abc)'(abd)'c] = [c(abc)'(abd)']$
$= \{[c(abc)'a]'[c(abc)'b]'d\} = \{[c(abc)'a]'ed\} = \{[ac(abc)']'ed\}$
$= \{[(aca)'(acb)'c]'ed\} = \{[(aac)'(acb)'c]'ed\} = \{[a(acb)'c]'ed\}$
$= \{[ca(acb)']'ed\} = \{[(caa)'(cac)'b]'ed\} = \{[(aac)'(cac)'b]'ed\}$
$= \{[a(cac)'b]'ed\} = \{[a(cca)'b]'ed\} = [(acb)'ed] = [d(acb)'e]$
$= \{(acb)'(acb)'[c(abc)'b]'\} = (acb)$, by 2, 1, A_3, A_5, A_3, A_5, A_3, A_5, A_3, 2, 1, A_3, A_5, A_3, 2, 1, A_3, 2, 1, A_3, 2, 1.

PROOF OF 4. By A_3, 3.

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* We shall use the symbols ( ), [ ], { } interchangeably to denote the fundamental ternary operation.

† I am indebted to H. S. Zuckerman for this proof.

In the rest of the proofs implicit use will be made of 1 and 4.

PROOF OF 5. $[a'(abc)'(a'b'c')'] = \{[a'(abc)'a']'[a'(abc)'b']'c'\}$ $= \{a'[a'b'(abc)']'c'\} = \{a'[(a'b'a)'(a'b'b)'c]'c'\} = [a'c'(b'a'c)']$ $= [(a'c'a')'(a'c'c)'b'] = [a'(a'c'c)'b'] = (a'a'b') = a$, by A_5, 2, A_5, A_4, A_5, 2, A_4, 2.

PROOF OF 6. $[a(abc)'(ab'c')'] = \{[a(abc)'a]'[a(abc)'b']'c'\}$ $= \{a[ab'(abc)']'c'\} = \{ac'[(ab'a)'(ab'b)'c]'\} = \{ac'[a(ab'b)'c]'\}$ $= [ac'(aac)'] = (ac'a) = a'$, by A_5, 2, A_5, 2, A_4, 2, 2.

PROOF OF 7. $(abc) = [ab(dd'c)'] = [(abd)'(abd')'c]$, by A_4, A_5.

PROOF OF 8. Put $(abd)' = p$, $(abd')' = q$, $(a'b'd)' = r$, $(a'b'd')' = s$.

Then

$$[d'(abc)'(a'b'c')'] = [d'(pqc)'(rsc')'] = \{[d'r(pqc)']'[d's(pqc)']'c'\}$$
$$= \{[(d'rp)'(d'rq)'c]'[(d'sp)'(d'sq)'c]'c'\},$$

by 7, A_5, A_5. But $(d'rq) = d$, $(d'sp) = d$ by 5, and $(d'sq) = d$ by 6. Hence the last expression reduces to

$$\{[cd'(d'rp)']'(cd'd')'c'\} = \{c'd'[cd'(d'rp)']'\} = [(c'd'c)'(c'd'd')'(d'rp)']$$
$$= [d'(c'd'd')'(d'rp)'] = [d'd'(d'rp)'] = d, \text{ by } 2, A_5, A_4, 2, 2.$$

PROOF OF 9. Suppose that for two fixed elements b, c, $(a'bc) = a$ for every element a. Then for $a = c$, $c = (c'bc) = b'$ by A_4.

PROOF OF 10. By 8, 9.

5. *Sufficiency and Necessariness of the Postulates.* The Whitehead-Huntington postulates leave undefined a *class* K and two *binary operations* $+$ and $\times$, and are the propositions Ia, Ib, $\cdots$, VI below. In postulates IIIa-IVb the condition *if the elements involved and their indicated combinations belong to* K is understood; in V the condition *if the elements* Z *and* u *of* IIa *and* IIb *exist and are unique* is understood.

POSTULATE Ia. *$a+b$ is in K whenever a and b are in K.*

POSTULATE Ib. *ab is in K whenever a and b are in K.*

POSTULATE IIa. *There is an element Z such that $a+Z=a$ for every element a.*

POSTULATE IIb. *There is an element u such that $au=a$ for every element a.*

POSTULATE IIIa. $a+b=b+a$.

POSTULATE IIIb. $ab=ba$.

POSTULATE IVa. $a+bc=(a+b)(a+c)$.

POSTULATE IVb. $a(b+c)=ab+ac$.

POSTULATE V. *For every element a there is an element $\bar{a}$ such that $a+\bar{a}=u$ and $a\bar{a}=Z$.*

POSTULATE VI. *There are at least two elements, a and b, in K such that $a\neq b$.*

We deduce Ia-VI from our postulates as follows:

Let u be *any* element in K. Then we may make the following definitions:

DEFINITION 2. $Z=u'$.

DEFINITION 3. $a+b=(abu)'$.

DEFINITION 4. $ab=(abZ)'$.

PROOF OF Ia. By Definition 3, A_2, Definition 1.

PROOF OF Ib. By Definition 4, A_2, Definition 1.

PROOF OF IIa. $a+Z=(aZu)'=(au'u)'=a$, by Definition 3, Definition 2, A_4.

PROOF OF IIb. $au=(auZ)'=(auu')'=a$, by Definition 4, Definition 2, A_4.

PROOF OF IIIa. $a+b=(abu)'=(bau)'=b+a$, by Definition 3.

PROOF OF IIIb. $ab=(abZ)'=(baZ)'=ba$, by Definition 4.

PROOF OF IVa. $a+bc=[a(bcZ)'u]'=[(abu)'(acu)'Z]'=(a+b)(a+c)$, by Definition 3, Definition 4, A_5, Definition 3, Definition 4.

PROOF OF IVb. $a(b+c)=[a(bcu)'Z]'=[(abZ)'(acZ)'u]'=ab+ac$, by Definition 3, Definition 4, A_5, Definition 3, Definition 4.

PROOF OF V. $a + a' = (aa'u)' = u$, by Definition 3, A_4. $aa' = (aa'Z)' = Z$, by Definition 4, A_4. Hence $a' = \bar{a}$.

PROOF OF VI. By A_1.

In the converse derivation we may of course assume all the theorems of Boolean algebra, since they follow from the Whitehead-Huntington postulates. Postulates A_1-A_5 may then be verified without any difficulty after defining (abc) by $a'b'+b'c'+c'a'$.

6. *Relation between Ternary and Binary Boolean Algebra. Derivation of DeMorgan's Formula.* We first prove the fundamental relation $(abc) = a'b'+b'c'+c'a'$.

PROOF. $(abc) = (a'b'c')' = [(a'b'Z)'(a'b'Z')'c']'$
$= \{[Z'(a'b'Z)'Z]'(a'b'Z')'c'\}' = \{[Z'(a'b'Z)'Z]'[a'b'(ZZ'Z')']'c'\}'$
$= \{[Z'(a'b'Z)'Z]'[Z'(a'b'Z)'(a'b'Z')']'c'\}'$
$= \{Z'(a'b'Z)'[Z(a'b'Z')'c']'\}' = \{Z'(a'b'Z)'[(Zc'a')'(Zc'b')'Z']'\}'$
$= \{(a'b'Z)'[(b'c'Z)'(c'a'Z)'u]'u\}' = a'b'+b'c'+c'a'$, by 10, 7, A_4, 2, A_5, A_5, A_5, and Definitions 2, 3, and 4.

We next observe that 10 is a generalization of DeMorgan's formula. Indeed,

$$(a + b)' = (abu) = (a'b'u')' = (a'b'Z)' = a'b'.$$

7. *Consistency and Independence of the Postulates.*

The consistency of postulates A_1-A_5 is shown by the following example.

EXAMPLE 1.0. $K = 1, 2$.

$c = 1$

	1	2
1	2	2
2	2	1

$c = 2$

	1	2
1	2	1
2	1	1

The independence proofs follow.

EXAMPLE 1.1. $K = 1$.

$c = 1$

	1
1	1

EXAMPLE 1.2. $K=1, 2$.

$c=1$

	1	2
1	3	3
2	3	3

$c=2$

	1	2
1	3	3
2	3	3

EXAMPLE 1.3. $K=1, 2$.

$c=1$

	1	2
1	2	2
2	1	1

$c=2$

	1	2
1	2	1
2	1	1

Postulate A_3 fails for $a=2$, $b=c=1$.

EXAMPLE 1.4. $K=1, 2$.

$c=1$

	1	2
1	1	1
2	1	1

$c=2$

	1	2
1	1	1
2	1	1

Postulate A_4 fails for $a=2$, $b=c=1$.

EXAMPLE 1.5.* $K=1, 2$.

$c=1$

	1	2
1	1	2
2	2	1

$c=2$

	1	2
1	2	1
2	1	2

Postulate A_5 fails for $a=c=1$, $b=d=e=2$.

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* The following alternate proof-systems, due in part to J. C. C. McKinsey, are of interest. Ex. 1.0. $K=1, 2$; $(abc)=[(a+b+c)(a+b+c+1)/2] \pmod 2$. Ex. 1.1. $K=1$; $(abc)=1$. Ex. 1.2. $K=1, 2$; $(abc)=3$. Ex. 1.3. $K=1, 2$; $(abc)=a$. Ex. 1.4. $K=1, 2$; $(abc)=1$. Ex. 1.5. $K=1, 2$; $(abc)=[a+b+c] \pmod 2$. A_5 fails for $a=c=d=e=1$, $b=2$.

SOME FORMULAS FOR FACTORABLE POLYNOMIALS IN SEVERAL INDETERMINATES†

BY LEONARD CARLITZ

1. *Introduction.* By a factorable polynomial‡ in the $GF(p^n)$ will be meant a polynomial in the indeterminates $x_1, \cdots, x_k$, which factors into a product of linear factors in some (sufficiently large) Galois field:

$$G \equiv G(x_1, \cdots, x_k) \equiv \prod_{j=1}^{m} (\alpha_{j0} + \alpha_{j1}x_1 + \cdots + \alpha_{jk}x_k).$$

It is frequently convenient to consider separately those G (of degree m) in which x_k^m (or any assigned x_i^m) actually occurs; we use the notation G^* to denote such a polynomial. In the case $k=1$, the polynomials G reduce to ordinary polynomials in a single indeterminate; in this case G and G^* are identical.

In this note we extend certain results§ for $k=1$ to the case $k>1$. For polynomials G^* the extensions may (roughly) be obtained by merely replacing p^n by p^{nk}; for arbitrary G the generalizations are not quite so simple.

2. *The μ-Function.* For G of degree m, we put $|G| = p^{nm}$; then

$$\zeta^*(w) = \sum_{G^*} \frac{1}{|G|^w} = (1 - p^{n(k-w)})^{-1}, \tag{1}$$

$$\begin{aligned} \zeta(w) &= \sum_{G} \frac{1}{|G|^w} \\ &= \{(1 - p^{n(1-w)})(1 - p^{n(2-w)}) \cdots (1 - p^{n(k-w)})\}^{-1}, \end{aligned} \tag{2}$$

the sums extending over *all* G^*, G, respectively.

Let $f(m)$ be the number of (non-associated) G of degree m, $f^*(m)$ the number of G^*; from the first of these formulas it follows that $f^*(m) = p^{nkm}$, and from the second, $f(m) = [k+m-1, m]p^{nm}$, where

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‡ Duke Mathematical Journal, vol. 2 (1936), pp. 660–670.

§ American Journal of Mathematics, vol. 54 (1932), pp. 39–50; this Bulletin, vol. 38 (1932), pp. 736–744.

$$(3)\qquad [k,s]=\frac{(p^{kn}-1)(p^{(k-1)n}-1)\cdots(p^{(k-s+1)n}-1)}{(p^n-1)(p^{2n}-1)\cdots(p^{sn}-1)}.$$

Taking the reciprocal of (1) and (2), we have

$$(4)\qquad \sum_{G^*}\frac{\mu(G)}{|G|^w}=1-p^{n(k-w)},$$

$$(5)\qquad \sum_{G}\frac{\mu(G)}{|G|^w}=\prod_{j=1}^{k}(1-p^{n(j-w)}),$$

where $\mu(G)$ is the Möbius function. From (4) it follows that

$$\sum_{\deg G^*=m}\mu(G)=\begin{cases}-p^{nk} & \text{for } m=1,\\ 0 & \text{for } m>1;\end{cases}$$

on the other hand, from (5) follows

$$\sum_{\deg G=m}\mu(G)=\begin{cases}(-1)^m[k,m]p^{nm(m+1)/2} & \text{for } m\leqq k,\\ 0 & \text{for } m>k,\end{cases}$$

where $[k, m]$ is defined by (3).

3. *The Divisor Functions.* If $\tau(G)$ denotes the number of divisors of G, then it is clear from (1) that

$$(6)\qquad \sum_{G^*}\frac{\tau(G)}{|G|^w}=(1-p^{n(k-w)})^{-2},$$

while from (2) it follows that

$$(7)\qquad \sum_{G}\frac{\tau(G)}{|G|^w}=\prod_{j=1}^{k}(1-p^{n(j-w)})^{-2}.$$

From (6) we have at once

$$\sum_{\deg G^*=m}\tau(G)=(m+1)p^{nmk}.$$

Similarly by means of (7), we may evaluate $\sum\tau(G)$, summed over all G of degree m:

$$\sum_{\deg G=m}\tau(G)=\sum_{m=i+j}[k+i-1,i][k+j-i,j]p^{nm}.$$

For the function $\sigma_t(G)=\sum|D|^t$, summed over all divisors of G, there are the formulas

$$(8)\quad \sum_{G} \frac{\sigma_t(G)}{|G|^w} = \zeta(w)\zeta(w-t), \qquad \sum_{G^*} \frac{\sigma_t(G)}{|G|^w} = \zeta^*(w)\zeta^*(w-t).$$

From the latter it is clear that

$$\sum_{\deg G^*=m} \sigma_t(G) = p^{nkm} \frac{p^{nt(m+1)} - 1}{p^{nt} - 1}.$$

The corresponding formula for $\sum\sigma_t(G)$, summed over *all* G of degree m, is not so simple in general. However, if $t=k$, the product $\zeta(w)\zeta(w-k)$ is itself a zeta-function, and thus we get from the first equation in (8)

$$\sum_{\deg G=m} \sigma_k(G) = [2k+m-1, m]p^{nm}.$$

4. *The ϕ-Functions.* Obviously, the Euler ϕ-function cannot be defined in terms of a reduced residue system. Instead we define $\phi_s(G)$ as the number of polynomials A of degree s such that $(A, G)=1$. For $k=1$, $s=\deg G$, $\phi_s(G)$ reduces to the Euler function (for polynomials in a single indeterminate). From the definition it is easily seen that

$$\sum_{s=0}^{\infty} \phi_s(G)p^{-nsw} = \sum_{(A,G)=1} |A|^{-w} = \zeta(w)\prod_{P|G}(1-|P|^{-w}),$$

and therefore, by equating coefficients of p^{-nsw},

$$(9)\qquad \phi_s(G) = \sum_{D|G}{}'\mu(D)f(s-d),$$

where $d=\deg D$, and the sum is over all divisors of degree $\leqq s$. For $s\geqq\deg G$, the sum is over all D; for $s=\deg G$, we shall omit the subscript, so that

$$(10)\qquad \phi(G) = \sum_{D|G} \mu(D)f(s-d),$$

summed over all divisors of G.

Similarly, $\phi_s^*(G)$ is the number of A^* of degree s such that $(A, G)=1$. Then

$$(11)\quad \phi_s^*(G) = \sum_{D|G}{}'\mu(D)f^*(s-d) = |G|^k \sum_{D|G}{}'\mu(D)\,|D|^{-k}.$$

Again for $s=\deg G$, we write simply $\phi^*(G)$, and we have

$$(12)\quad \phi^*(G) = |G|^k \sum_{D|G} \mu(D)\,|D|^{-k} = |G|^k \prod_{P|G} (1 - |P|^{-k}),$$

where P denotes a typical irreducible divisor of G.

For $\phi^*(G)$ the sum function (taken over G^*) is quite simple. Substituting from (12), we find

$$(13)\quad \sum_{G^*} \frac{\phi^*(G)}{G^w} = \sum_{D^*} \frac{\mu(D)}{|D|^w} \sum_{E^*} \frac{|E|^k}{|E|^w} = \frac{\zeta^*(w-k)}{\zeta^*(w)}$$

$$= (1 - p^{n(k-w)}) \sum_{j=0}^{\infty} p^{nj(2k-w)},$$

and therefore

$$(14)\quad \sum_{\deg G^* = m} \phi^*(G) = p^{2nmk} - p^{nk(2m-1)} \qquad \text{for } m \geqq 1.$$

In the second place, we may extend the sum in the left member of (13) over all G:

$$\sum_{G} \frac{\phi^*(G)}{|G|^w} = \sum_{D} \frac{\mu(D)}{|D|^w} \sum_{E} \frac{|E|^k}{|E|^w} = \frac{\zeta(w-k)}{\zeta(w)},$$

from which follows

$$\sum_{\deg G = m} \phi^*(G) = \sum_{m=i+j} (-1)^i [k, i][k+j-1, j] p^{n(k+1)i} p^{ni(i+1)/2}.$$

For $\phi(G)$ the formulas corresponding to (13) and (14) are

$$(15)\quad \sum_{G^*} \frac{\phi(G)}{|G|^w} = \sum_{D^*} \frac{\mu(D)}{|D|^w} \sum_{E^*} \frac{f(e)}{|E|^w} = \frac{\zeta(w-k)}{\zeta^*(w)},$$

and

$$\sum_{\deg G^* = m} \phi(G) = [k+m-1, m] p^{nm(k+1)}$$

$$- [k+m-2, m-1] p^{n(mk+m-1)}.$$

Finally, if the sum on the left of (15) be taken over all G,

$$\sum_{G} \frac{\phi(G)}{|G|^w} = \sum_{D} \frac{\mu(D)}{|D|^w} \sum_{E} \frac{f(e)}{|E|^w} = \frac{1}{\zeta(w)} \sum_{e=0}^{\infty} \frac{f^2(e)}{p^{new}},$$

and therefore

$$\sum_{\deg G = m} \phi(G) = \sum_{m=i+j} (-1)^i [k, i][k+j-1, j]^2 p^{ni(i+1)/2} p^{2nj}.$$

We remark that more general ϕ-functions may be defined, and the corresponding sum functions constructed exactly as above. For brevity the formulas are omitted.

5. *The q-Functions.* We now consider polynomials L that are not divisible by the eth power of an irreducible. The number of L of degree m will be denoted by $q_e(m)$; the number of L^* by $q_e^*(m)$. For the latter function, it is evident that

$$\sum_{m=0}^{\infty} q_e^*(m)p^{-nmw} = \prod_{P^*}(1 + |P|^{-w} + \cdots + |P|^{-(e-1)w}) = \frac{\zeta^*(w)}{\zeta^*(ew)},$$

where P^* denotes a typical irreducible starred polynomial. Then

$$q_e^*(m) = \begin{cases} p^{nmk} & \text{for } m < e, \\ p^{nmk} - p^{nk(m-e+1)} & \text{for } m \geqq e. \end{cases} \tag{16}$$

On the other hand, since

$$\sum_{m=0}^{\infty} q_e(m)p^{-nmw} = \frac{\zeta(w)}{\zeta(ew)}$$

$$= \sum_{i=0}^{\infty}[k + i - 1, i]p^{ni}p^{-nwi}\sum_{j=0}^{k}(-1)^j[k, j]p^{nj(j+1)/2}p^{-newj},$$

we have in place of (16),

$$q_e(m) = \sum_{m=i+ej}(-1)^j[k + i - 1, i][k, j]p^{ni}p^{nj(j+1)/2}. \tag{17}$$

Next, let

$$Q(m) = \prod_{\deg L=m} L, \quad Q^*(m) = \prod_{\deg L^*=m} L^*.$$

If we put

$$D_s = D_s(x_1, \cdots, x_k) = |x_i^{p^{sj}}|, \quad (i, j = 0, \cdots, k),$$

where x_0 is replaced by 1, and

$$\Delta_s = \frac{D_s(x_1, \cdots, x_k)}{D_s(x_1, \cdots, x_{k-1})},$$

then for

$$F_s^*(m) = \Delta_m\Delta_{m-1}^{p^{nk}} \cdots \Delta_1^{p^{nk(m-1)}},$$

we may show, exactly as in the case† $k=1$, that

$$\prod_{s=0}^{h}\{Q^*(se+r)\}^{p^{nk(h-s)}} = F_1^*(he+r)\{F_e^*(h)\}^{-ep^{nkr}}$$

$$= R_e(he+r), \tag{18}$$

say, where $0 \leqq r < e$. From (18) follows at once

$$Q_e^*(m) = R_e(m)\{R_e(m-e)\}^{-p^{nk}}. \tag{19}$$

For $Q(m)$ the generalization is not entirely satisfactory. In place of (18) we have

$$\prod_{s=0}^{h}\{Q_e(se+r)\}^{f(h-s)} = \frac{F(he+r)}{\prod_{j=0}^{h-1} D_{h-j}^{ef(je+r)}},$$

where

$$F(m) = D_m D_{m-1}^{f(1)} \cdots D_1^{f(m-1)}$$

(the product of all polynomials of degree m). However, there seems to be no simple formula like (19) for $Q_e(m)$.

DUKE UNIVERSITY

† See p. 743 of the paper in this Bulletin referred to above.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

STATE COLLEGE, PENNSYLVANIA, SUMMER MEETING AND COLLOQUIUM, September 7–10, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than August 3. Abstracts received by the Secretaries not later than July 9 will appear in the July issue of the Bulletin. Professor John von Neumann will deliver a series of colloquium lectures on *Continuous geometry*. By invitation of the Program Committee, Professor Hassler Whitney will give an address entitled *Topological properties of differentiable manifolds*.

NEW YORK CITY, October 30, 1937.

Abstracts must be in the hands of Associate Secretary T. R. Hollcroft, 531 West 116th St., New York, N. Y., not later than October 2. Abstracts received by the Secretaries not later than September 3 will appear in the September issue of the Bulletin. By invitation of the Program Committee, Professor D. J. Struik will deliver an address entitled *The application of tensor analysis to problems of electrical engineering*.

PASADENA, CALIFORNIA, November 27, 1937.

Abstracts are due by October 30, 1937.

R. G. D. RICHARDSON, *Secretary of the Society*.

Articles for insertion in the BULLETIN should be addressed to the Managing Editor, D. R. CURTISS, Northwestern University, Evanston, Ill. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees and the annual dues of members of the Society (see this BULLETIN, p. 322, May, 1930; and the List of Officers and Members, September, 1936, p. 67) are payable to the Treasurer of the Society, Professor P. A. Smith, Low Memorial Library, 531 West 116th St., New York, N. Y.

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